Problem 1 Find the domain of the following function:

$$f(x) = \frac{1}{\sqrt{x-3}} \cdot \frac{x^2+1}{x^2-25}$$

Solution: The domain is $(3,5) \cup (5,+\infty)$.

Detailed explanation: For $\sqrt{x-3}$ to be defined, x must be greater than or equal to 3. That is,

$$\text{Domain}(\sqrt{x-3}) = [3,+\infty)$$

.

Since Domain $\frac{1}{x} = \mathbb{R} - \{0\}$, $\sqrt{x-3}$ can't be 0 and so Domain $\frac{1}{\sqrt{x-3}} = (3,+\infty)$. Furthermore, $\text{Domain}(x^2+1) = \mathbb{R}$, as it is a polynomial, and

$$\text{Domain}\left(\frac{1}{x^2-25}\right) = \{x \,:\, x^2 - 25 \neq 0\},$$

that is, the set of all x such that $x^2 - 25 \neq 0$. This set is $x \neq \pm 5$, also denoted $\mathbb{R} - \{-5, 5\}$.

The domain of f is the intersection of the domains of the two functions of which it is a product. So the domain of f is $x > 3$ such that $x \neq \pm 5$. We get $(3,\infty) - \{5\} = (3,5) \cup (5,\infty)$.

Additional problem(s): Find the domain of f in these cases:

- $f(x) = \dfrac{1}{\sqrt{|x-3|-7}}$

  Solution: $(-\infty,-4) \cup (10,\infty))$ **Detailed explanation:**Don't forget the absolute value in $|x-3|$.

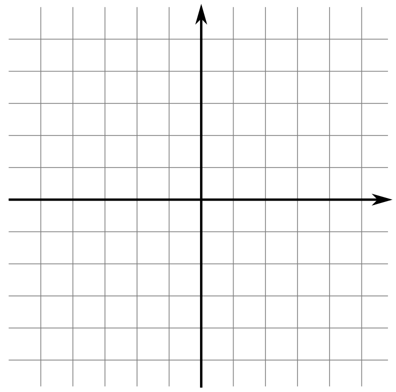
- $f(x) = \dfrac{1}{x^2+1} \cdot \dfrac{\sqrt{9-x^2}}{x^2-4}$

- $f(x) = \sin\left(\dfrac{\log\left(\sqrt{3-x}\right)}{3+x}\right)$

Problem 2

a. Sketch the graph of the following piecewise-defined function:

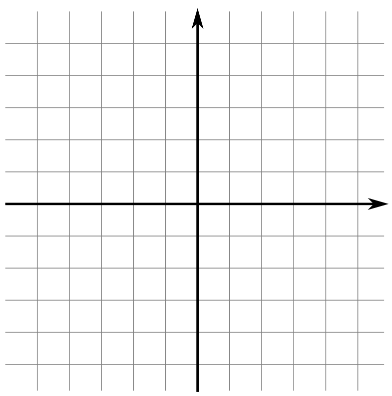
$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x < 4; \\ x - 1, & \text{if } x \geq 4. \end{cases}$$

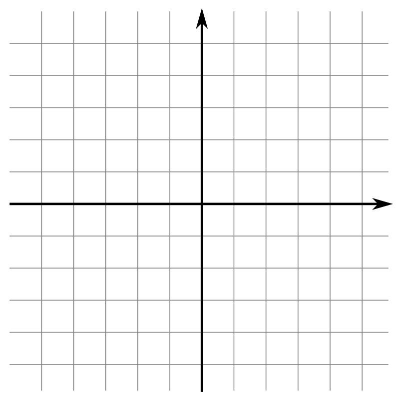


b. Is this function continuous?

Solution: This function is **not** continuous, since $\lim\limits_{x\to 4^-} = 2$, and $\lim\limits_{x\to 4^+} = 3$.

c. Sketch the graph of $2f(x-2)$ on the left [**4 points**], and graph of $f^{-1}(x)$ on the right





Solution: the graph on the left is the graph of f shifted 2 unites to the right and stretched out 2 times vertically. The graph on the left is the graph of f reflected through the line $y = x$.

Problem 3 You and your friend attended a show of the famous band Klein Four. After the show ended, you bought 7 records, and your friend bought 4. Your total expenses were \$125, and your friend's expenses totaled \$80.

The ticket prices are the same, and all records cost the same. Build a model of the cost of attendance of the show. Interpret the slope and y-intercept of the function you get.

Solution: the cost of attendance, as a function of the number of records purchased, is $f(x) = 20 + 15x$. The slope is the price of a record. The y-intercept is the ticket price.

Detailed explanation:you bought $7 - 4 = 3$ more records than your friend, and spent $\$125 - \$80 = \$45$, so the cost of a record is $\$45/3 = 15$, and the cost of the ticket is $\$80 - 4 \cdot \$15 = \$125 - 7 \cdot \$15 = \$20$.

Problem 4 Let f, g be functions whose domain is $(-\infty, +\infty)$. The values of f and g for some values of x are given in the table below:

x	1	2	3	4	5
f(x)	3	3	3	3	2
g(x)	1	6	10	5	3

a. What is $f \circ g(5)$?

Solution: $f \circ g(5) = f(g(5)) = f(3) = 3$.

b. What is $g \circ f(5)$?

Solution: $g \circ f(5) = g(f(5)) = g(2) = 6$.

c. Is f one-to-one?

Solution: No, it is not: $f(1) = f(2)$, for example.

d. Is g one-to-one?

Solution: We can't tell given this data. While g does not take the same value on $\{1, 2, 3, 4, 5\}$, we don't know what goes on elsewhere. It could be that $g(1) = g(1.1)$, for example.

e. **Additional problem(s):**

- Let h be a function whose domain is $\{1, 2, 3, 4, 5\}$, and $h(x) = g(x)$ wherever defined. Is h one-to-one?

  Solution: Yes! The function h does not take the same value twice on its domain. So it is one-to-one, and has an inverse.

Problem 5 The population of fruit flies around the bananas in your kitchen (that you totally forgot about) doubles every 36 hours. There are 10 flies buzzing happily now.

a. How many fruit flies will you have in three days?

Solution: Three days is 72 hours. You'll have $10 \cdot 2^{72/36} = 10 \cdot 4 = 40$ fruit flies.

b. What is the function $f(x)$ that describes the number of flies you have after x hours?

Solution: $f(x) = 10 \cdot 2^{x/36}$.

Problem 6 Constant hyperinflation in the country of Artztozka decreases the value of savings in Artztozkan Grubles (as measured in USD) by a factor of 0.8 every 18 months. You just purchased \$1000 worth of Arztztozkan Grubles.

a. What will be the value of your investment in 2 years?

Solution: $\$1000 \cdot 0.8^{2/1.5}$.

b. Arztotzkan (never-changing) president swore to resign if an investment into Grubles depreciates to 10% of its initial value in USD. How many years will pass before you are able to call him out on that fake promise?

c. $1.5 \cdot \log_{0.8}(1/10) = 1.5 \cdot \dfrac{\log(1/10)}{\log(0.8)} = 15.47$. Not that he is expected to resign even then.

Problem 7 You measure velocity of a falling water balloon, in m/s, in terms of time passed, in s, since you dropped it from the rooftop of the mathdepartment towards your favorite math professor.

t, s	1	1.5	2	2.5	3
V, m/s	14	20	23	28	35

It seems like the constant pull of gravity yields a linear model. Use linear regression to determine the model.

Solution: $V(t) = at + b$ would be the general equation; you should get coefficients close to $f(x) = 10t + 4$ (not these exact numbers, though).

Problem 8 The x coordinate of the the tip of the second clock hand, in cm, is given by the equation $x(t) = 15\sin(2\pi t/60)$.

a. Find the average velocity V_{ave} on the following intervals I:

1. $I = [30, 32]$.

   Solution: $V_{\text{ave}} = \dfrac{x(32) - x(30)}{2} = ...$

2. $I = [30, 30.5]$, $V_{\text{ave}} = ...$
3. $I = [30, 30.1]$, $V_{\text{ave}} = ...$
4. $I = [30, 30.01]$

   Solution: $V_{\text{ave}} = \dfrac{x(30.01) - x(30)}{0.01} = ...$

b. What the instantaneous horizontal velocity at $t = 30$s?

Solution: the numbers you get will be approaching $-pi/2$.

Detailed explanation: This is equation describes the continuous movement of the hand tip at constant speed; it travels $15 \cdot 2\pi$ cm over 60 seconds, so is always moving at a speed of $30\pi/60 = \pi/2$ cm/2. At $t = 30$, the hand tip is pointing down, and the direction of motion is to the left.

Problem 9 Let $f(x) = x^2$. Calculate (or approximate to 4 decimal places) the following limit:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

for $a = 7$.

Solution:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{(a+h)^2-a^2}{h}\\
&= \lim_{h\to 0}\frac{a^2+2ah+h^2-a^2}{h}\\
&= \lim_{h\to 0}\frac{2ah+h^2}{h}\\
&= \lim_{h\to 0} 2a+h\\
&= 2a.
\end{aligned}$$

So the limit is $2a = 14$.

Problem 10 Calculate (or approximate to 4 decimal places) the following limit:

$$\lim_{x\to\infty}\frac{2x^3+5x+1}{3x^4+5x^2+7}.$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty}\frac{2x^3+5x+1}{3x^4+5x^2+7} &= \lim_{x\to\infty}\frac{2+5/x^2+1/x^3}{3x+5/x+7/x^3}\\
\lim_{x\to\infty}\frac{2}{3x}\\
\lim_{x\to\infty} 0.
\end{aligned}$$

Detailed explanation: Divide the top and bottom of the fraction by x^3, and notice that

$$\lim_{x\to\infty}\frac{1}{x^n}=0$$

for all $n>0$.

Problem 11 Let $f(x)=\sin(x)$, where x is in degrees, so $f(30)=0.5$.

a. For the following intervals $I_a=(0.5-a,0.5+a)$, find intervals J_b of the form $(30-b,30+b)$ so that f takes values in I_a on the interval J_b you find.

   1. $a=0.02$, $I_a=(0.52,0.48)$;

      Solution: $J_b=(29,31)$ (try it!).

   2. $a=0.001$, $I_a=(0.501,0.499)$;

      Solution: $J_b=(29.99,30.01)$.

   Detailed explanation: Experiment by trying values of x near $x=30$ to find b such that $0.5-a<\sin(30-b)<\sin(30+b)<0.5+a$. This works because $\sin(x)$ is monotonous (increasing) on a stretch around $x=30$.

b. Let $g(x)$ be defined as follows:

$$g(x) = \begin{cases} f(x), x \neq \dfrac{\pi}{4}; \\ 0, x = \dfrac{\pi}{4}. \end{cases} .$$

Find the limit $\lim_{x \to 4} g(x)$, or state that it does not exist.

Solution:

$$\lim_{x \to \pi/4^-} g(x) = \lim_{x \to \pi/4^-} f(x) = \frac{1}{\sqrt{2}}$$
$$\lim_{x \to \pi/4^+} g(x) = \lim_{x \to \pi/4^+} f(x) = \frac{1}{\sqrt{2}}$$
$$g\left(\frac{\pi}{4})\right) = 0$$

The left/right limits are equal, but not to the value of g at $\pi/4$.

Therefore, the limit **does not exist**.