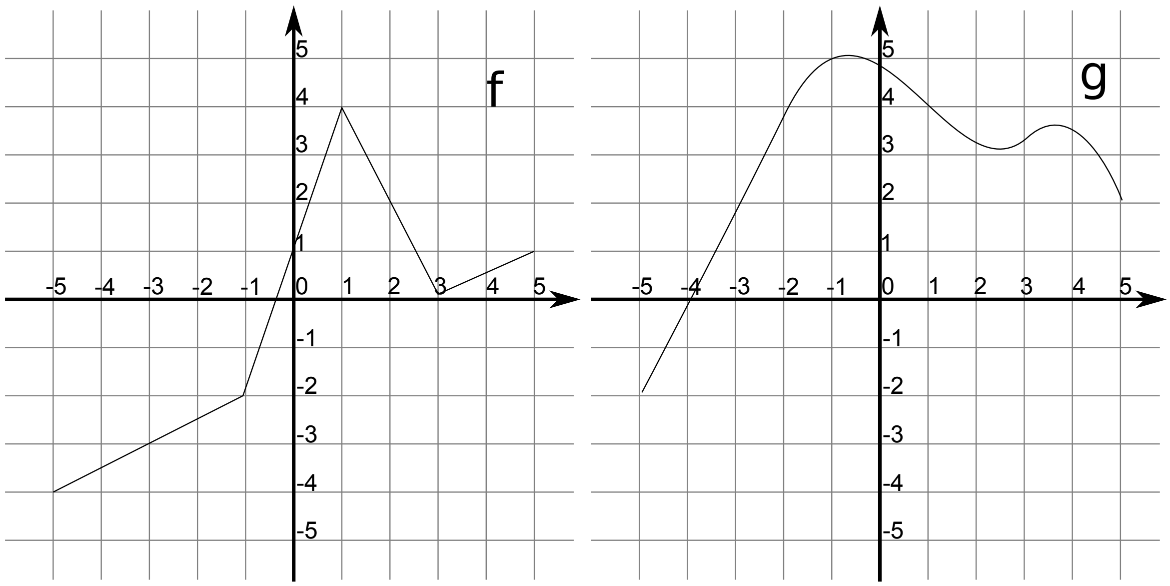
# MATH 131:501 Midterm Exam 2

# PRACTICE PROBLEMS

**Problem 1:** Let $f, g$ be functions whose graphs are provided below.




1. Where do $f'$, $g'$ exist?

   **Solution:** $f'(x)$ exists everywhere $f$ is defined except for $x \in \{-1, 1, 3\}$. $g'(x)$ exists everywhere $g$ is defined.

2. Where is $f(x)$ increasing/decreasing? Where is $f'(x)$ positive/negative? Same questions for $g$.

   **Solution:** $f$ is increasing on $(-5, 1)$ and $(3, 5)$; $f'$ is positive on those intervals.

3. Find $f'(-3)$ and $g'(-3)$

   **Solution:** $f'(-3) = 1/2$, and $g'(-3) = 2$, as these are the respective slopes.

4. If $F(x) = f(x)/g(x)$, $F'(-3) =$

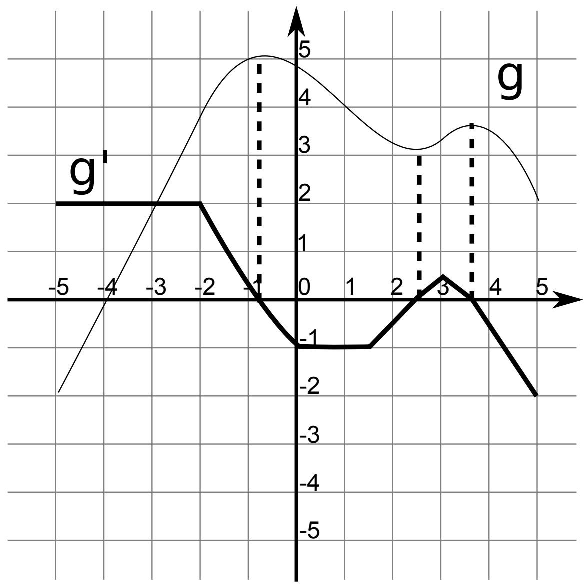
   **Solution:** $F'(-3) = \dfrac{f'(-3)g(-3) - g'(-3)f(-3)}{g(-3)^2} = \dfrac{\frac{1}{2} \cdot 2 - 2 \cdot -3}{2^2}$

5. Estimate $g'(1.5)$ from the graph. Use it to find the equation of the tangent line.

   **Solution:** The slope is approximately $-1$, so $g(1.5) \approx -1$. Since $g(1.5) \approx 3.5$, the tangent equation is $-(x - 1.5) + 3.5$.

6. Sketch $f'(x)$ [**3 points**] and $g'(x)$ [**3 points**].

**Solution:** For $f$, the graph of $f'$ is piecewise-constant, with values 1/2, 3, $-2$ and 1/2 on corresponding intervals. The approximate sketch of $g'$ is below.



**Problem 2:** Let $f(x) = \dfrac{1}{x+1}$. Using the **defintion of derivative only**, find the deriviative of $f(x)$. **Note:** show all work. No credit will be given for just an answer. **You may not use the power rule**.

**Solution:**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+1+h} - \frac{1}{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+1)-(x+1+h)}{(x+1+h)(x+1)}}{h} \\
&= \lim_{h\to 0} \frac{(x+1)-(x+1+h)}{h(x+1+h)(x+1)} \\
&= \lim_{h\to 0} \frac{-h}{h(x+1+h)(x+1)} \\
&= \lim_{h\to 0} \frac{-1}{h(x+1+h)(x+1)} \\
&= \frac{-1}{(x+1)^2}.
\end{aligned}$$

**Problem 3:** Let $f(x) = \ln(x)$. Find $F'(x)$ using **only** the chain rule and the fact that $(e^x)' = e^x$.

**Solution:** by definition, $e^{\ln(x)} = x$. By the chain rule, $\dfrac{d}{dx}e^{\ln(x)} = e^{\ln(x)}(\ln(x))'$.

Differentiating both sides, we get

$$\begin{aligned}
\frac{d}{dx}e^{\ln(x)} &= \frac{d}{dx}x \\
\Rightarrow e^{\ln(x)}(\ln(x))' &= 1 \\
x(\ln(x))' &= 1 \\
(\ln(x))' &= \frac{1}{x} \\
F'(x)' &= \frac{1}{x}.
\end{aligned}$$

**Problem 4:** After taking an exam, you throw your favorite instructor through the window. The altitude of your instructor, in meters, as a function of time, in seconds, is given by the function $f(t) = 6 - 5(t-1)^2$. How fast (in the vertical direction) did you throw your favorite instructor? How fast is your instructor going at the moment of impact with the ground?

**Solution:** the initial velocity is $f'(0) = 2 \cdot -5(0-1) = 10\dfrac{m}{s}$. The moment of impact is when $f(t) = 0$; solving for $t$, we find that $t \approx 2.0954$, and $f'(t) \approx -10.9545$; that is, $10.9545\dfrac{m}{s}$ downwards.

**Problem 5:** Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. The values of $f(x), f'(x), g(x), g'(x)$ for some values of $x$ are given in the table below.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 3 | 5 | 4 | 6 | 2 |
| $g(x)$ | 4 | 1 | 3 | 2 | 6 | 5 |
| $f'(x)$ | 1 | 5 | 3 | 2 | 4 | 11 |
| $g'(x)$ | 5 | 4 | 3 | 11 | 7 | 6 |

Using the rules of differentiation, compute the following. **Justify your work!**

1. $F(x) = f(x) \cdot g(x)$. $F'(2) =$
2. $F(x) = f(x)/g(x)$. $F'(2) =$
3. $F(x) = g(f(x))$. $F'(3) =$
4. $F(x) = e^{g(x)}$. $F'(5) =$
5. $F(x) = 2f(x) + 3g(x)$. $F'(5) =$
6. $F(x) = \ln(f(x) \cdot g(x))$. $F'(1) =$
7. $F(x) = \dfrac{\sin(f(x))}{\cos(g(x)+1)}$. $F'(3) =$.

**Sample solutions:** To get **1**, note that $F'(2) = f'(2)g(2) + f(2)g'(2)$ by the product rule, and so $F'(2) = 5 \cdot 1 + 3 \cdot 4 = 17$.

To get **4**, note that $F'(5) = e^{g(5)}g'(5) = e^7 \cdot 6$ by the chain rule.

Other parts are done in a similar way.

**Problem 6:** Find the following derivatives:

1. $\dfrac{d}{dx}\left(x^{1/2} + x^{1/3} + 7\right) = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{3}x^{-2/3}$ (power rule)
2. $\dfrac{d}{dx}\left(\dfrac{\cos(x)}{\sqrt{x^2+1}}\right) = \dfrac{-\sin(x)\sqrt{x^2+1} - \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \cdot \cos(x)}{x^2+1}$ (quotient, power, chain rule)
3. $\dfrac{d}{dx}\left(\dfrac{\sin(x^3)}{e^x}\right) = \dfrac{3x^2\cos(x^3)e^x - e^x\sin(x^3)}{e^{2x}}$
4. $\dfrac{d}{dx}(\ln(\sin(x)\cos(x))) = \dfrac{\sin'(x)\cos(x) + \cos'(x)\sin(x)}{\sin(x)\cos(x)} = \dfrac{\cos^2(x) - \sin^2(x)}{\sin(x)\cos(x)}$.

   Note that we could get the same result by noting that

   $$\ln(\sin(x)\cos(x)) = \ln(\sin(x)) + \ln(\cos(x)).$$

   Then $\dfrac{d}{dx}(\ln(\sin(x)) + \ln(\cos(x))) = \dfrac{\cos(x)}{\sin(x)} + \dfrac{-\sin(x)}{\cos(x)} = \dfrac{\cos^2(x) - \sin^2(x)}{\sin(x)\cos(x)}$.

**Problem 7:** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function on the real line. Match the limits below to their values.

1. $\lim_{a\to 0} \frac{f(x+a)-f(x)}{a}$
2. $\lim_{a\to 0} \frac{f(2a)-f(-2a)}{4a}$
3. $\lim_{a\to 0} \frac{f(a)-f(0)}{2a}$
4. $\lim_{a\to 0} \frac{f(0)-f(2a)}{2a}$
5. $\lim_{a\to 0} \frac{f(a)-f(0)}{a}$
6. $\lim_{a\to 0} \frac{f(x+a)-f(x-a)}{2a}$
7. $\lim_{a\to 0} \frac{f(-a)-f(a)}{-a}$

a. $f'(0)$
b. $-f'(0)$
c. $f'(0)/2$
d. $2f'(0)$
e. $f'(x)$
f. $-f'(x)$
g. $f'(x)/2$
h. $2f'(x)$

**Key:**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| e | a | c | b | a | e | d |

**Notes:** By definition, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, which is the expression in **1** (except with the variable $h$ renamed to $a$); thus **1-e**.

Similarly, $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(0)-f(0)}{h}$, and so **5-a**.

But these are not the only expressions for a derivative: the general idea is that

$$f'(x) = \lim \frac{\text{difference in } \textbf{value} \text{ of f}}{\text{difference in } \textbf{argument} \text{ of f}} = \lim_{b,a\to x} \frac{f(b)-f(a)}{b-a}$$

as both $b$ and $a$ approach $x$ (and thus the difference in the argument, $b-a$, approaches 0).

For that reason, $f'(x) = \lim_{a\to 0} \frac{f(x+a)-f(x-a)}{2a}$ as well (so **6-e**), and **2-a:** $f'(0) = \lim_{a\to 0} \frac{f(2a)-f(-2a)}{4a}$

To get other limits, note that

$$\lim_{a\to 0} \frac{f(a)-f(0)}{2a} = \frac{1}{2} \lim_{a\to 0} \frac{f(a)-f(0)}{a} = \frac{1}{2} f'(0),$$

so **3-c**; similarly,

$$\lim_{a\to 0} \frac{f(0)-f(2a)}{2a} = -\lim_{a\to 0} \frac{f(2a)-f(0)}{2a} = -f'(0),$$

so **4-b**, and

$$\lim_{a\to 0} \frac{f(-a)-f(a)}{-a} = 2 \lim_{a\to 0} \frac{f(a)-f(-a)}{2a} = 2f'(0),$$

so **7-d**.