MATH 131:501 Exam 3 practice

Your name: _____

April 25, 2016

Problem 1: Let $f(x)$ be a differentiable function on $\mathbb{R}$. Some values of $f'(x)$ are given in the table below.

x	0	1	2	3	4
$f'(x)$	0	5	6	3	1

Assuming $fm(0) = 0$, approximate $f(4)$.

Solution: we have, by the FTC:

$$f(4) - f(0) = \int_0^4 f'(x)dx.$$

We can approximate this integral with a left Riemann sum:

$$\{x_0, x_1, \ldots, x_4\} = \{0, 1, 2, 3, 4\}$$

$$\begin{aligned}
LRS &= \sum_{n=0}^{3} f'(x_n)(x_{n+1} - x_n) \\
&= f'(x_0)(x_1 - x_0) + f'(x_1)(x_2 - x_1) + f'(x_2)(x_3 - x_2) + f'(x_3)(x_4 - x_3) \\
&= 0 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 + 3 \cdot 1 \\
&= 14.
\end{aligned}$$

We could also use a right Riemann sum:

$$\{x_0, x_1, \ldots, x_4\} = \{0, 1, 2, 3, 4\}$$

$$\begin{aligned}
RRS &= \sum_{n=1}^{4} f'(x_n)(x_{n+1} - x_n) \\
&= f'(x_1)(x_1 - x_0) + f'(x_2)(x_2 - x_1) + f'(x_3)(x_3 - x_2) + f'(x_4)(x_4 - x_3) \\
&= (5 + 6 + 3 + 1) \cdot 1 \\
&= 15.
\end{aligned}$$

Either of the two is an approximation of $f(4) - f(0)$.

We can get a better approximation by averaging the two:

$$f(4) - f(0) \approx \frac{LRS + RRS}{2} = 14.5.$$

Since $f(0) = 0$, $f(4) \approx 14.5$.

Problem 2: Let $f(x) = e^{x^2}$.

1. Approximate the area under the graph of f on $[0, 2]$ by a **right** Riemann sum with 7 equally-spaced rectangles.

   Solution: the length of each sub-interval is $\dfrac{2}{7}$, so the partition (i.e. the endpoints) is:

   $$\{x_0, x_1, x_2, \ldots, x_7\} = \left\{0, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \ldots, \frac{14}{7} = 2\right\}$$

   All sub-intervals have the same width, $\Delta x = \dfrac{2}{7}$. Therefore,

   $$\begin{aligned}
   RRS &= \sum_{n=1}^{7} f(x_i)\Delta x \\
   &= f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_7)\Delta x \\
   &= f\left(\frac{2}{7}\right)\cdot\frac{2}{7} + f\left(\frac{4}{7}\right)\cdot\frac{2}{7} + \ldots + f\left(\frac{14}{7}\right)\cdot\frac{2}{7} \\
   &= \frac{2}{7}\left(e^{\left(\frac{2}{7}\right)^2} + e^{\left(\frac{4}{7}\right)^2} + \ldots + e^{\left(\frac{14}{7}\right)^2}\right) \\
   &= \underline{\hspace{4in}} \quad \text{(whatever this equals to - do it!)}.
   \end{aligned}$$

2. Approximate the area under the graph of f on $[0, 2]$ by a **left** Riemann sum with 7 equally-spaced rectangles.

   Solution: the left Riemann sum is nearly the same, except we use the left endpoints:

   $$\begin{aligned}
   RRS &= \sum_{n=0}^{6} f(x_i)\Delta x \\
   &= f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_6)\Delta x \\
   &= \frac{2}{7}\left(e^{\left(\frac{2}{7}\right)^2} + e^{\left(\frac{4}{7}\right)^2} + \ldots + e^{\left(\frac{14}{7}\right)^2}\right) \\
   &= \underline{\hspace{4in}} \quad \text{(do it!)} \\
   &= RRS - \frac{2}{7}e^{\left(\frac{14}{7}\right)^2} + \frac{2}{7}e^{\left(\frac{2}{7}\right)^2} \quad \text{(check this!)}.
   \end{aligned}$$

   Notice that we can save the effort and get RRS from LLS by subtracting the area of the last rectangle and adding the first. We can do this because all rectangles have the same width.

3. Which of these is an over / underestimate? Justify your answer.

   LRS is an underestamite since f is increasing: $f'(x) = 2xe^{x^2} > 0$ for $x > 0$.

   Similarly, RRS is an overestimate.

Problem 3: Find the area under the curve $f(x) = \sin(x)$ for x between 0 and π.

Solution: Since $\sin(x)$ is positive on $(0, \pi)$,

$$A = \int_0^\pi \sin(x)dx = -\cos(x)|_0^\pi = -\cos(pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

Problem 4: the velocity of a particle at time t, in seconds, is given by $f(t) = t^3$, in m/s. Find the net change in position...

1. when t goes from -1 to 1

   Solution: if $p(t)$ is the function that gives the position, then $f(t) = p'(t)$, and the net change is given by:

   $$p(1) - p(-1) = \int_{-1}^1 p'(t)dt = \int_{-1}^1 f(t)dt = \int_{-1}^1 t^3 dt = \frac{t^4}{4}\Big|_{-1}^1 = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0.$$

   So the net change is 0 - the object returned to its initial position.

2. ... when t goes from 0 to 2

   Solution: following the above,

   $$p(2) - p(0) = \frac{t^4}{4}\Big|_0^2 = \frac{16}{4} = 4.$$

3. if $p(t)$ is the function that gives the position, what is $p'(3)$?

   Solution: $p'(3)$ is the velocity at $t = 3$. But we are given the velocity: it is $f(t)$. So $p'(3) = f(3) = 3^3 = 27\text{m/s}$.

Problem 5: Let $F(T)$ be the area under the curve of $f(t) = t^2$ from $t = 0$ to $t = T$. For T between 0 and 4, where is F changing the fastest?

Solution: the change in area under the graph of f between 0 and T is proportional to the value of the function at the boundary, that is, $f(T)$.

This can be also seen from the FTC:

$$\frac{d}{dT}F(T) = \frac{d}{dT}\int_0^T f(t)dt = f(T).$$

The area is changing the fastest when $|F'(T)|$ has the largest magnitude.

Since $|F'(T)| = |f(T)|$, we just need to find the maximum of $|f(T)|$ for T between 0 and 4.

But $f(x)$ is increasing and is positive, so $|f(T)| = f(T)$, and its maximum is at the boundary (at $T = 4$).

Therefore, for T between 0 and 4, the area under the graph of f from 0 to T is changing the fastest when $T = 4$.

Specifically, it is increasing. Intuitively, when you move T from 4 to 4.1 you get way more increase in area under the graph than when you move T from 1 to 1.1.

Problem 6: Find the indefinite integrals:

$$\int 5x^3 + \cos(x) + 3dx = \frac{5}{4}x^4 + \sin(x) + 3x + C$$

$$\int \sin(x^2)2xdx = -\cos(x^2) + C$$

$$\int \frac{1}{x} + \frac{1}{x^2} + \sqrt{x} = \ln(x) - \frac{1}{x} + \frac{2}{3}x^{\frac{3}{2}} + C$$

Problem 7: What is the largest (by area) isosceles triangle with perimeter 30?

Solution: let x be the base of the triangle; then the length of a side is $(30 - x)/2$. Draw an altitude to the base, forming a right-angled triangle with hypotenuse $(30 - x)/2$ and a side $x/2$. By the Pythagorean theorem, the altitude has length

$$\sqrt{\left(\frac{30 - x}{2}\right)^2 - \left(\frac{x}{2}\right)^2} = \frac{\sqrt{900 - 60x}}{2}.$$

From this equation (and from the triangle inequality), x must be between 0 and 15 (otherwise, the two other sides are too short and won't meet - and you'll be taking a square root of a negative expression). We get this bound by solving $\sqrt{900 - 60x} \geq 0$ and common sense (side length can't be negative).

The area (which depends on x) is $A(x) = \frac{1}{2}$base $\times$ height, so

$$A(x) = \frac{1}{2}x \cdot \frac{\sqrt{900 - 60x}}{2} = \frac{x}{4}\sqrt{900 - 60x}.$$

Thus, we are looking for a global maximum of A on the interval $[0, 15]$.

Since $A(0) = A(15) = 0$, the maximum is a local maximum, and therefore happens at a zero of a derivative; that is, when $A'(x) = 0$. So we solve:

$$A'(x) = 0$$
$$\frac{d}{dx}\frac{x}{4}\sqrt{900 - 60x} = 0$$
$$\sqrt{900 - 60x} = \frac{60x}{2\sqrt{900 - 60x}}$$
$$900 - 60x = 30x$$
$$900 = 90x$$
$$10 = x.$$

So $x = 10$ is a critical point, and

$$A(10) = \frac{10}{4}\sqrt{900 - 60 \cdot 10} = 25\sqrt{3},$$

for a triangle with base 10 and two sides of length $(30 - 10)/2 = 10$.

This must be a local maximum, since $A(0) < A(10)$, and so the area was increasing as x went from 0 to 10 (and there are no other critical points).

Alternatively, you can check that $A'(9.9) > 0$ and $A'(10.1) < 0$; or you can calculate and see that $A''(10) < 0$, meaning that it's a local maximum.

The local maximum is the global one, since A is equal to 0 on the boundary. So the answer is - the largest-area triangle is an equilateral one with side 10 and area $25\sqrt{3}$.

Problem 8: Let $f(x) = 4x^7 - 35 \cdot 7x^4 + 27 \cdot 8 \cdot 28x$ be a function on the interval $[1.5, 4]$.

1. Find all the critical points, classify them as local minima/maxima.

   Solution: we solve $f'(x) = 0$:

$$\frac{d}{dx}4x^7 - 35 \cdot 7x^4 + 27 \cdot 8 \cdot 28x = 0$$
$$28x^6 - 35 \cdot 28x^3 + 27 \cdot 8 \cdot 28 = 0$$
$$x^6 - 35x^3 + 27 \cdot 8 = 0$$
$$(x^3 - 27)(x^3 - 8) = 0$$
$$x^3 = 27 \text{ or } x^3 = 8$$
$$x = 3 \text{ or } x = 2.$$

   So 3 and 2 are the critical points.

   Now $f''(x) = 28 \cdot 6x^5 - 35 \cdot 3 \cdot 28x^2 = 84(2x^5 - 35x^2)$; we have $f''(2) = 84(2 \cdot 2^5 - 35 \cdot 2^2) = 84(64 - 140) < 0$. You can check that $f''(3) > 0$. Therefore, 2 is a local maximum, and 3 is a local minimum.

2. Where is f increasing / decreasing?

   We check the sign of $f'(x)$ around critical points:

$$f'(1.75) = \underline{\hspace{2cm}}$$
$$f'(2.5) = \underline{\hspace{2cm}}$$
$$f'(3.5) = \underline{\hspace{2cm}}$$

   Since $f'(1.75) > 0$, $f'(2.5) < 0$, and $f'(3.5) > 0$, it follows that

$$f'(x) > 0 \text{ for } x \in [1.5, 2];$$
$$f'(x) < 0 \text{ for } x \in [2, 3];$$
$$f'(x) > 0 \text{ for } x \in [3, 4].$$

Therefore,

$$f \text{ is increasing on } [1.5, 2];$$
$$f \text{ is decreasing on } [2, 3];$$
$$f \text{ is increasing on } [3, 4].$$

3. Where is f concave up/down?

   Solution: we solve for $f''(x) = 0$ to check where f changes concavity:

   $$f''(x) = 84(2x^5 - 35x^2) = 0$$
   $$2x^5 - 35x^2 = 0$$
   $$2x^5 = 35x^2$$
   $$x^3 = \frac{35}{2} \text{(can divide by } x^2 \text{ since } x \in [1.5, 4] \text{ and so } x \neq 0)$$
   $$x = \sqrt[3]{\frac{35}{2}}.$$

   So f changes concavity once on $[1, 4]$. We know $f''(2) < 0$, and $f''(3) > 0$; so f is concave down on $[1, \sqrt[3]{35/2}]$ and is concave up on $\sqrt[3]{35/2}, 4]$.

4. where are the points of inflection?

   Solution: $\sqrt[3]{\frac{35}{2}}$ is the only point of inflection.

5. What is the global minimum / maximum of f on $[1.5, 4]$?

   Solution: we have found all the critical points already, so we just check them and the endpoints:

   $$f(1.5) = \underline{\hspace{2cm}}$$
   $$f(2) = \underline{\hspace{2cm}}$$
   $$f(3) = \underline{\hspace{2cm}}$$
   $$f(4) = \underline{\hspace{2cm}}$$

   It turns out that f has a global minimum at $x = 3$, and a global maximum at $x = 4$ (since these values of x give the smallest/largest values in the list, respectively).

Problem 9:   Let $f(x)$ be a differentiable function on $\mathbb{R}$. Given that $f''(x) = 12x + 2$, $f(0) = 1$, and $f(1) = 3$, find f.

Solution: to find $f(x)$, we find $f'(x)$ first:

$$f'(x) = \int f''(x) \, dx = \int 12x + 2dx = 6x^2 + 2x + C.$$

Note that we found $f'(x)$ only up to a constant, C, whose value we don't know. Now we can find $f(x)$:

$$f(x) = \int f'(x)\,dx = \int 6x^2 + 2x dx = 2x^3 + x^2 + Cx + D.$$

Again, if we know $f'(x)$, we can only find $f(x)$ up to a constant. Since this constant has nothing to do with our previous computation, we give it a different name and call it D.

To find the values of constants, we plug in the initial conditions:

$$f(0) = D = 1$$
$$f(1) = 2 + 1 + C + D = 3$$

From these equations, $D = 1$, and $C = -D = -1$. Therefore,

$$f(x) = 2x^3 + x^2 - x + 1.$$