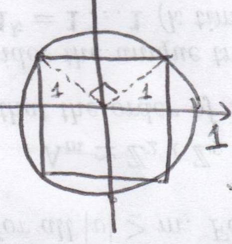
Extra credit 1

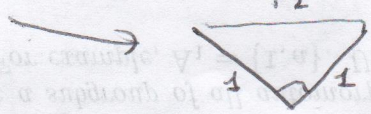
Goal: approximate π using elementary operations ($+, -, \cdot, /, \sqrt{\ }$)

Approach: approximate circle with a square, octagon, ..., 2^n-gon, getting more & more precise approximations. Obtain π in the limit.

Step 1:



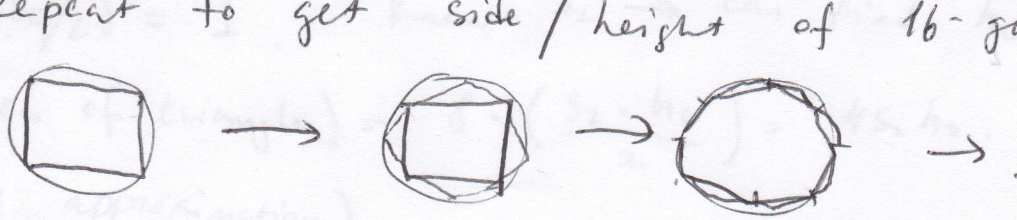
A square inside the unit circle has side $s_1 = \sqrt{1^2+1^2} = \sqrt{2}$



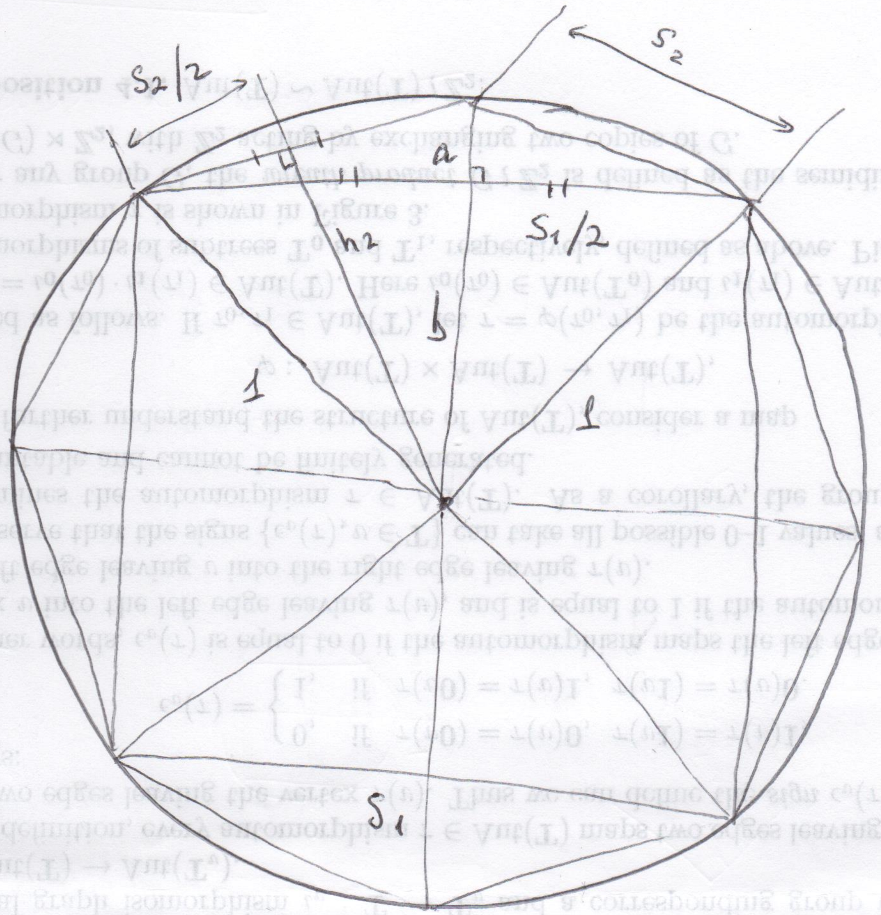
Its area is $(\sqrt{2})^2 = 2$, which is our first approximation

In subsequent steps, this approximation is improved upon.

Idea:
- If we know the side of the square, can figure out the side of the octagon
- Can compute height, too $\to$ get area
- Repeat to get side / height of 16-gon, 32-gon, ...



Step 2: From square to octagon



S_1 = side of square

$(S_1 = \sqrt{2})$

S_2 = side of octagon

dea: Solve for S_2 in terms of S_1

By pythagorean thm, $S_2^2 = a^2 + (S_1/2)^2$

also $a + b = 1$

and by pythagorean thm, $b^2 + (S_1/2)^2 = 1$

know: S_1 unknowns: a, b, S_2 ; 3 equations, 3 unknowns → can solve

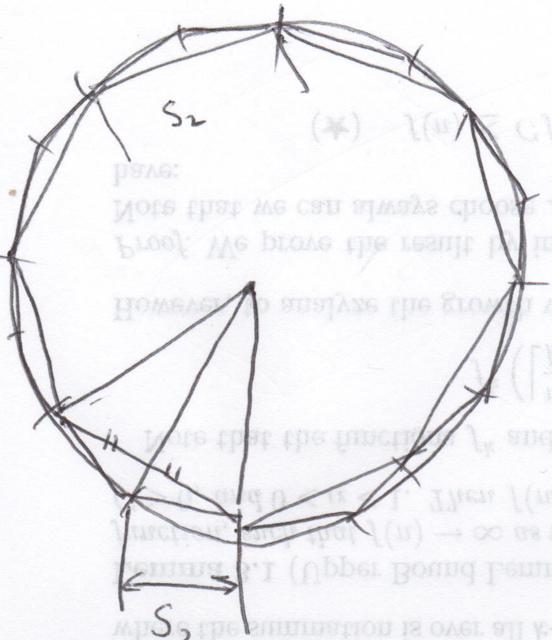
find area, drop altitude h_2 to a side of octagon from the center. Then by the Pythagorean thm,

$h_2^2 + (S_2/2)^2 = 1$. Know S_2 → can find h_2.

rea $= 8 \cdot$ (area of triangle) $= 8 \cdot \left(\frac{S_2 \cdot h_2}{2}\right) = 4 S_2 h_2$.

$\overset{\shortparallel}{A_2}$ (second approximation)

Step 3! rinse & repeat



The formula for s_3 in terms of s_2 is the same as the formula for s_2 in terms of s_1.

So you can get s_3, s_4, s_5, s_6, ... without having to solve equations again (but you may do so if you're stuck).

Then $A_3 = 16$ (area of $\triangle$) $= 16\left(\frac{h_3 \cdot s_3}{2}\right)$,

and where $h_3^2 + \left(\frac{s_3}{2}\right)^2 = 1$.

A_4 is going to be a decent approximation of π.

And in the limit, $\pi = \lim_{n \to \infty} A_n$

Can you guess what the formula will look like for that limit?

Addendum: infinite formulas

Formulas don't have to be finite.

Example: let $a_1 = 1$

$$a_2 = \frac{1}{1+1}$$

$$a_3 = \frac{1}{1+\frac{1}{1+1}}$$

$$a_4 = \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}$$

$$a_5 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}$$

$$\vdots$$

Then ~~lim~~ $\lim_{n\to\infty} a_n = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$

With your calculator, try computing a_{10}, a_{15}, a_{20}

(it's fast if you use the fact that $a_n = \frac{1}{1+a_{n-1}}$)

In the limit, you'll get $\frac{\sqrt{5}-1}{2} \approx 0.618\ldots$

With the extra credit, the goal is to get an infinite formula for π.

Extra credit 2.

Math history project

- Use book, Ch. 5.4, p. 374

"Newton, Leibniz & the invention of Calculus"

Things to address in your report:

- Why did Newton/Leibniz discover calculus? What problems were they solving?
- Leibniz's notation vs. Newton's notation
- Which problems became easy with the invention of calculus (e.g. Zeno's paradox, specific area problems, etc.)
- What kind of calculus problems in Newton's time were difficult to solve (e.g. 3-body problem).

Use any sources.