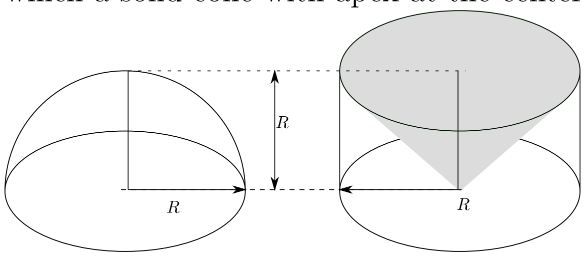
Name: ______________________________ **Section:** 201 (starts at 08:00)
202 (starts at 09:35)

Justify your answers.

Consider a half-sphere of radius R, and a solid cylinder of radius R and height R, out of which a solid cone with apex at the center of the bottom face has been removed.



We will compute the volume of the sphere by looking to cross-sections parallel to the bases of these solids.

1. (2 points) Find the cross-section area of the half-sphere at height h from the base (as a function of h).

   Solution: The cross-sections are disks of radius $r = \sqrt{R^2 - h^2}$ and area $\pi r^2 = \pi(R^2 - h^2)$.

2. (2 points) Find the cross-section area of the second solid at height h (as a function of h).

   Solution: The cross-section is an annulus (a disk with a hole). The radius of the outer circle is R, the radius of the inner cicle is h, and the area is $\pi(R^2 - h^2)$.

3. (1 point) What is the volume of the second solid?

   Solution: The volume is $V(\text{cylinder}) - V(\text{cone}) = \pi R^2 \cdot R - \pi R^2 \cdot R \cdot \frac{1}{3} = \frac{2}{3}\pi R^3$.

4. (1 point)Apply Cavalieri's principle to get the volume of the half-sphere, as a function of R.

   Solution: By Cavalieri's principle, since the cross-sections are the same, so are the volumes. The volume of the hemisphere is thus $\frac{2}{3}\pi R^3$.

5. (4 points) Find the volume of the sphere of radius 1 using the method of cylindrical shells.

   Solution: the shells have radius $r = x$ and height $h = \sqrt{1 - x^2}$, so

$$
\begin{aligned}
V &= \int_{-1}^{1} 2\pi \cdot r(x) \cdot h(x)\ dx \\
&= \int_{-1}^{1} 2\pi \cdot x \cdot \sqrt{1 - x^2}\ dx \\
&= 2\int_{x=0}^{x=1} 2\pi \cdot x \cdot \sqrt{1 - x^2}\ dx
\end{aligned}
$$

Letting $u = 1 - x^2$, $du = -2x\ dx$ we make the subsitution:

$$\begin{aligned} V &= 2 \int_{u=u(0)=1}^{u=u(1)=0} -\sqrt{u}\ du \\ &= 2 \int_0^1 \sqrt{u}\ du \\ &= 2 \frac{2}{3} u^{3/2} |_0^1 \\ &= \frac{4}{3}, \end{aligned}$$

which is consistent with the answer for the previous problem.