A Paper of Interest,

or What do Credit Cards Have to Do with Natural Logs?

by Roman Kogan

The subject of loans and monthly payments is not interesting only if there is no interest on the loan, or if the interest is simple (you pay back a fixed percentage of what you loan). Neither of these cases is realistic; in the real world, the interest is compounded, so you end up paying interest on interest. This is bad for our pockets, but good for discussing mathematics behind it.

1 Don't worry, we'll be discrete

Often, the interest is broken up into even parts and applied at regular time intervals. This is known as *discrete compounding*.

For a common scenario, consider a loan of P dollars over N years with APR (annual percentage rate) r, with interest accrued monthly. This means that without any payments, the loan will increase by a factor of $(1 + r/12)$ each month: we take $1/12$ of the yearly interest and apply it every month.

As a practical example, suppose you have a credit card with 24% APR (not uncommon for student cards), so $r = 0.24$, and you have a balance of \$1000 in the beginning of the year. Applying $r/12 = 2\% = 0.02$ interest each month means that if you don't make any payments and there are no extra fees, in the end of the year you will owe

$$\begin{aligned} \$1000 \cdot \underbrace{(1+0.02)\cdot(1+0.02)\cdot\ldots\cdot(1+0.02)}_{12 \text{ times}} &= \$1000 \cdot (1+0.02)^{12} \\ &= \$1268.24 \text{ (rounded to a cent)}, \end{aligned}$$

which is greater than \$1240 you would expect to pay if the interest was accrued annually. *Side note: yes, it's actually written in small font in the credit card agreement.*

Now suppose you loan P dollars and want to pay it off after N years (that is, $12N$ months), making a fixed payment of M dollars each month. So after the first month, you will owe $P \cdot \left(1 + \frac{r}{12}\right) - M$, after the accumulation of interest and payment (we assume the interest is accrued before payment). After the second month, you pay M again and owe

$$\left(P \cdot \left(1 + \frac{r}{12}\right) - M\right) \cdot \left(1 + \frac{r}{12}\right) - M = P\left(1 + \frac{r}{12}\right)^2 - M\left(1 + \frac{r}{12}\right) - M.$$

We now can start building a table of how much you owe:

Time	Debt
Initially	P
After 1 month	$P\left(1+\frac{r}{12}\right)-M$
After 2 months	$P\left(1+\frac{r}{12}\right)^2-M\left(1+\frac{r}{12}\right)-M$
After 3 months	$P\left(1+\frac{r}{12}\right)^3-M\left(1+\frac{r}{12}\right)^2-M\left(1+\frac{r}{12}\right)-M$
...	...

Now the pattern should be clear. After N years, that is, $12N$ months go by, you will owe

$$P\left(1+\frac{r}{12}\right)^{12N}-M\left(1+\frac{r}{12}\right)^{12N-1}-M\left(1+\frac{r}{12}\right)^{12N-2}-\ldots-M\left(1+\frac{r}{12}\right)-M.$$

At this point, you want the debt to be paid off, so we write

$$P\left(1+\frac{r}{12}\right)^{12N}-M\left(1+\frac{r}{12}\right)^{12N-1}-M\left(1+\frac{r}{12}\right)^{12N-2}-\ldots-M\left(1+\frac{r}{12}\right)-M=0. \tag{1}$$

This is an equation in one unknown, M (P, N and r are given) . To solve it, we need to rewrite this expression in a more tractable form. We start by rewriting it like this:

$$\frac{P}{M}\left(1+\frac{r}{12}\right)^{12N}=\left(\left(1+\frac{r}{12}\right)^{12N-1}+\left(1+\frac{r}{12}\right)^{12N-2}+\ldots+\left(1+\frac{r}{12}\right)+1\right) \tag{2}$$

The right-hand side of this equation is now a geometric series. Recall the following formula:

$$\begin{aligned}(a-1)(a^{n-1}+a^{n-2}+\ldots+a^2+a+1)&=(a^n-1)\\ \Rightarrow (a^{n-1}+a^{n-2}+\ldots+a^2+a+1)&=\frac{a^n-1}{a-1}\end{aligned}$$

Verify it! After you multiply, the extra terms will cancel out. This formula allows us to deal with the right-hand side of equation (2), with $a=\left(1+\frac{r}{12}\right)$ and $n=12N$:

$$\begin{aligned}\frac{P}{M}\left(1+\frac{r}{12}\right)^{12N}&=\frac{\left(1+\frac{r}{12}\right)^{12N}-1}{\left(1+\frac{r}{12}\right)-1}\\ &=\frac{\left(1+\frac{r}{12}\right)^{12N}-1}{\frac{r}{12}}.\end{aligned}$$

Finally, we obtain the answer:

$$\begin{aligned}M&=P\left(1+\frac{r}{12}\right)^{12N}\frac{\frac{r}{12}}{\left(1+\frac{r}{12}\right)^{12N}-1}\\ \Rightarrow M&=P\frac{\frac{r}{12}}{1-\left(1+\frac{r}{12}\right)^{-12N}}\end{aligned}$$

The formula above has the form you find in Gilat's Matlab Manual (Gilat assumes r is in percent, and has to divide by 100 to get it right). The total amount you pay is

$$12N \cdot M = P \frac{Nr}{1 - \left(1 + \frac{r}{12}\right)^{-12N}}$$

With the example above ($r = 24\% = 0.24$, $N = 1$), your monthly payment would be \$94.56, and you'll pay \$1134.72 in total on a \$1000 loan.

2 Curiouser and Curiouser: Continuous Compounding

So, from the previous section you saw that applying the interest to your debt at small even intervals allows the bank to increase your debt due to compounding (compared to accruing the interest once per year).

You might be asking: *how much* can the bank gain in this way, while keeping the APR fixed? It seems like accruing the interest more often benefits the bank. What if the interest was accrued every day? Every hour? Every *second*?

The bad news is that the more periods there are, the more you end up paying. More on that later, but here is a tabulation of debt on a loan of \$1000 at 24% APR after one year for various numbers of periods (rounded to a dollar), assuming no payments are made:

# periods	Debt
1	\$1240
2	\$1254
6	\$1265
12	\$1268
24	\$1269
100	\$1270
1000	\$1271
10000	\$1271

The good news is that there is a limit to the madness. For example, the table above suggests that the debt won't increase above \$1272 no matter how often the interest is applied. This is why banks often settle for 12 periods per year. Our goal now is to figure out what exactly is going on here.

To start, let's see what happens in the absence of payments (or, if you reverse the scenario, in absence of withdrawals from a savings account). If r is the interest rate, and n is the number of compounding periods per year, the interest accrued after each period is $\left(1+\frac{r}{n}\right)$, and the total amount after 1 year is

$$P_n = P\underbrace{\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)\dots\left(1+\frac{r}{n}\right)}_{n\text{ times}} = P\left(1+\frac{r}{n}\right)^n.$$

To proceed further, we need to expand $(1+r/n)^n$. Recall[1] the binomial coefficients formula:

$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^2+\ldots+\binom{n}{n}x^n, \text{ where}$$
$$\binom{n}{1} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot(k-2)\ldots 3\cdot 2\cdot 1}.$$

Applying this with $x = r/n$, we get:

$$P_n = P\left(1+\binom{n}{1}\frac{r}{n}+\binom{n}{2}\frac{r^2}{n^2}+\ldots+\binom{n}{n}\frac{r^n}{n^n}\right)$$

This is a polynomial in r. Let c_k be the coefficient of r^k in P_n/P. Then

$$\begin{aligned}
P_n/P &= 1+c_1r+c_2r^2+\ldots+c_nr^n;\\
c_k &= \binom{n}{k}\frac{1}{n^k}\\
&= \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k\ k!}\\
&= \frac{1}{k!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\ldots\cdot\frac{n-k+1}{n}\\
&= \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right).
\end{aligned}$$

We can extract a lot of information from this expression. First, notice that $(1-c/n)$, with $n = 1, 2, 3, \ldots$, is an *increasing* sequence for all $c > 0$. Since c_k is a product of such terms and $1/k!$, it follows that c_k increases as n increases. This means, P_n increases as n increases, as it becomes a sum of larger and larger terms. This formally proves our observations about P_n.

But we also get an upper bound on P_n/P (that is, a bound on how much the bank can make from increasing the number of periods).

Notice that for $c > 0$, $1 > (1-c/n)$ - obviously, and $k! > 2^{k-1}$ for $k \geq 0$ (a bit harder). Thus we have

$$c_k = \frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) \ \leq\ \frac{1}{k!} \ \leq\ \frac{1}{2^{k-1}}.$$

[1]Look up binomial coefficients in Wikipedia if you have trouble recalling what they are about!

Substituting into $P_n/P = 1 + c_1 r + \ldots + c_n r^n$, together with $r = 1$, we get

$$\begin{aligned} P_n/P &\leq 1 + r + \frac{r^2}{2} + \frac{r^3}{4} + \ldots + \frac{r^n}{2^{n-1}} \\ &= \leq 3. \end{aligned}$$

That is, with a 100% APR, your debt will, at worst, *triple* yearly. That's a relief!

Now to deal with other values of r, observe that

$$\begin{aligned} \left(1 + \frac{r}{n}\right)^n &= \left(1 + \frac{1}{n/r}\right)^n \\ &= \left(\left(1 + \frac{1}{n/r}\right)^{n/r}\right)^r \\ &\leq \left(\left(1 + \frac{1}{N}\right)^N\right)^r \quad \text{for any } N > n/r \\ &\leq 3^r, \end{aligned}$$

since we have shown that $(1 + 1/n)^n \leq 3$ for *all* n.

To summarize, this shows that no matter how often the interest is compounded, the debt increases by *at most* a factor of 3^r each year.

More importantly, we demonstrated that P_n/P is an *increasing, bounded* sequence for all values of r. It is a fundamental property[2] of real numbers that such (increasing, bounded) sequences must have a *least* upper bound. In calculus terms, this means that the sequence P_n/P *converges to a limit* (that depends on r). So,

$$\lim_{n\to\infty} P_n/P = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n \text{ exists.}$$

This allows us to define a function

$$\Phi(r) = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n,$$

which measures the percentage of the initial debt amount you have to pay back after a year at APR equal to r, assuming the interest was compounded infinitely often - that is, *continuously.*

To actually compute $\Phi(r)$, we can simply plug large values of n into $(1 + r/n)^n$. But there is a better way, which also gives more insight. Recall that the coefficients in the polynomials P_n/P were of the form

$$c_k = \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right).$$

[2]Unimaginatively, but aptly called the Least Upper Bound property

But $(1 - c/n) \to 1$ as $n \to \infty$, so $c_k \to \frac{1}{k!}$ as $n \to \infty$, and thus

$$\Phi(r) = 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \dots$$

We *know* the sum on the right-hand side converges because $\Phi(r)$ is well-defined. This gives another way to compute $\Phi(r)$ and approximate it. Cutting of the sum after a first few terms allows us to say that for small r,

$$\Phi(r) \approx (1 + r) + \frac{r^2}{2}.$$

That is, your effective interest rate is about $r^2/2$ more than the nominal one.

But wait, there is more![3] It turns out that $\Phi(r)$ has a much shorter expression (although not necessarily one that's better for computation). The same trick we used to find a bound works with limits:

$$\begin{aligned}
\Phi(r) &= \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n \\
&= \lim_{n\to\infty} \left(\left(1 + \frac{1}{n/r}\right)^{n/r}\right)^r \\
&= \lim_{N\to\infty} \left(\left(1 + \frac{1}{N}\right)^{N}\right)^r \quad \text{by substituting } N = n/r \\
&= \left(\lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^{N}\right)^r \\
&= \Phi(1)^r.
\end{aligned}$$

So, once you get $\Phi(1) = 1 + 1 + 1/2! + 1/3! + \dots$, you get $\Phi(r) = \Phi(1)^r$. Mathematicians found this so useful, they decided to give the constant $\Phi(1)$ a name. You might have seen it in the wild under the name e, Euler's constant, or the base of the natural logarithm. For the function $\Phi(x)$, people write $\exp(x)$ or simply e^x.

And so the journey begins. There are many questions that can be asked: what does all of this have to do with logarithms? Where else does the function $\exp(x)$ appear, and what other uses does it have? And so on. There is enough questions to write a book on[4]. All these questions can not be answered here, even if the margins were wider.

What **can**[5] be answered here is the question that brought us to this point: what should the monthly payment be? To answer it, note that under continuous compounding, the interest accrued each month is $\exp(r/12)$ (why?). Use this as a starting point, and use a computation similar to the one in the first

[3] If you call 1-800-GET-MATH within the next 20 minutes, you'll get $\Phi(r)$ as a solution of a differential equation *for free!*

[4] Which some people did, e.g. "'E: The Story of a Number"' by Eli Maor

[5] ...but would not.

section to get the answer. Compute the monthly payment for the example $r = 24\%$, $P = 1000$, $N = 1$, and compare it with the number in the end of Section 1.

As the tradition goes, the details are left as an exercise to the reader. The margins are all yours.