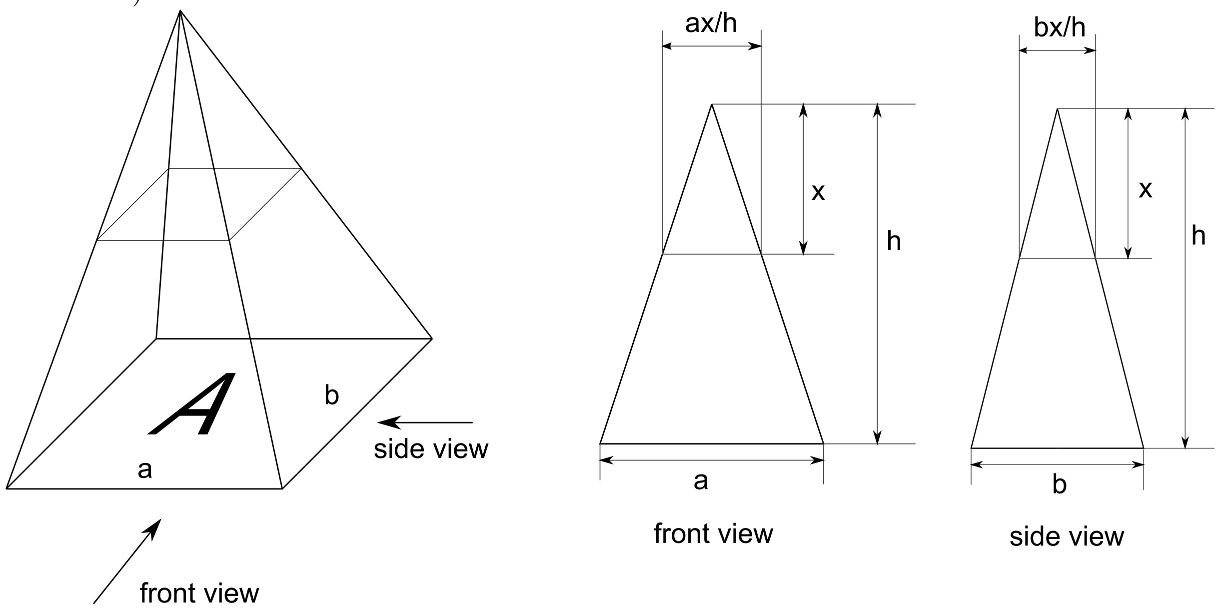
Homework 3

Homework: Find the volume of a solid cone whose base has area A, and whose vertex is distance h away from the plane of the base. Express your answer in terms of A and h. Justify all steps.
Clarification: In this problem, you *cannot* assume that the base is round. It can be any shape bounded by a continuous curve.
Definition: Let B be a subset of a plane, and v be not in the plane. A *solid cone* with base B and vertex v is the set of points lying on segments starting at v and ending in a point in B.
Solution: first, we solve the problem for the base being a rectangle with sides a and b (whose area is $A = ab$).



Cross-sections with planes parallel to the base are rectangles (think why). What side lengths do they have? Looking at the solid from the front (see front view above), by similar triangles we obtain that the front-facing side of the cross-section at distance x from the vertice has length ax/h. Similarly, the othe side has length bx/h. Thus, the slice at distance x away from the vertex has area

$$A(x) = \left(a\frac{x}{h}\right)\left(b\frac{x}{h}\right) = ab\left(\frac{x}{h}\right)^2 = A\left(\frac{x}{h}\right)^2, \tag{1}$$

since $ab = A$. Or simply observe that **the area of similar rectangles is proportional to the square of the similarity ratio**.
Now the volume is an easy integral: the slices are between 0 to h, and

$$V = \int_0^h A\left(\frac{x}{h}\right)^2 = A\frac{x^3}{3h^2}\Big|_0^h = A\frac{h^3}{3h^2} = \frac{Ah}{3}. \tag{2}$$

So, the answer for rectangular base is $V = \frac{1}{3}Ah$. What about other shapes?
Observe that the last equality in equation 1 holds for *any* shape. That is, if shapes S_1 and S_2 are similar with similarity ratio k, then

$$\frac{A(S_1)}{A(S_2)} = k^2.$$

So for any shape with area A, the slice at distance x from the vertex has area $A\left(\frac{x}{h}\right)^2$. The integral to get the volume stays the same (as in eqn. 2), giving $Ah/3$.

Intuition: Any shape can be filled with non-overlapping rectangles so that the total area of the rectangles approaches the area of the shape (this is the idea behind integration!). Think about drawing the shape on millimeter graph paper, and gettings its area by counting the squares inside it.

If you double the size of the shape, each side of each rectangle doubles, and the area quadruples. Similarly, if you scale by a factor of k, the area of each rectangle (and thus the shape) increases by a factor of k^2.

Another approach: consider a shape S. Let $A(S)$ denote the area of the shape, $C(S,p)$ denote a cone with base S and vertex p, $V(C(S,p))$ denote the volume of that cone.

Fill the shape S with rectangles $R_1, R_2, R_3, \ldots$. Then

$$\begin{aligned}
V(C(S,p)) &= V(C(R_1 \cup R_2 \cup R_3 \cup \ldots, p)) \\
&= V(C(R_1,p)) + V(C(R_2,p)) + V(C(R_3,p)) + \ldots \\
&= \frac{1}{3}hA(R_1) + \frac{1}{3}hA(R_2) + \frac{1}{3}hA(R_3) + \ldots \\
&= \frac{1}{3}h\Big(A(R_1) + A(R_2) + A(R_3) + \ldots\Big) \\
&= \frac{1}{3}hA(S) \\
&= \frac{1}{3}hA.
\end{aligned}$$