Solutions

1. $\lim_{n\to\infty} a_n$ is a number L such that for all $\epsilon > 0$ there exists a number N such that for $k > N$, $|a_k - L| < \epsilon$, whenever such number exists (see Ch. 10.1, def. 2).

   In other words, it is a number such that any open interval containing L also contains all but *finitely many* terms of the sequence.

   Very informally speaking, the limit exists and equals L if we can get arbitrarily close to L by going far enough in the sequence. This is discussed in the book (Ch. 10.1, def. 1).

   Note: although this definition appears late in the book, it is *used* in Chapter 6 in the defintions of area and the definite integral.

2. We say $\lim_{x\to x_0} f(x) = L$ if for all $\epsilon > 0$ there exists a number δ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon|$.

   In other words, any open interval containing L contains an image of an open interval around x_0.

   An intuitive approach taken in the book is by defining limits as x approaches x_0 on the left/on the right, and stating that if they are equal to a number L, that number is the limit. This can be made precise by considering *all* sequences a_n approaching x_0, and seeing if the corresponding $f(a_n)$ converge to the same limit L.

   If the limit exists, as in above, and, in addition, $f(x_0) = L$, we sat that f is **continuous** at x_0. In the book, look in Chapter 2.2, definitions 1, 2 and 3 for an intuitive introduction, and Chapter 2.4 definition 2 for a discussion of a precise definition.

3. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, whenever this limit exists. See Chapter 2.7, definitions 1 and 2. A geometric definition is: $f'(x)$ is the slope of the tangent to the graph of f at $(x, f(x))$.

4. $\int f(x)dx$ is a function $F(x)$ such that $F'(x) = f(x)$. See Chapter 6.4 definiton 9.

5. $\int_a^b f(x)dx$ is the signed area between the graph of the function $f(x)$ and the x-axis bounded by the vertical lines $x = a$ and $x = b$ (the area below the x-axis is counted as negative).

   More formally, it can be defined as the limit of an approximation of this area by rectangles:

   $$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x_k,$$

   where $\Delta x = (b-a)/N$ and $x_k = a+k\Delta x$. A more general definition is by summing over arbitrary partitions. See Chapter 6.2, definition 2, as well as Chapter 6.3. definition 3.

A word on notation

We use the greek letter Σ for sum, since it is the Greek letter S. The integral is also a sum, and so we use the stretched-out letter S for the integral (notation due to Leibniz).

Leibniz thought of the integral as a sum of infinitely many infinitely small numbers: by letting Δx become very small, he wrote it as dx, and $\sum f(x_k)\Delta x$ became $\int f(x)dx$. Rigorously defining infinitesimals like dx is hard (yet possible, which was done in 1960's) - so calculus was not a mathematically rigorous discipline until the late 1800's. People used calculus, since it is clear enough intuitively and gives a lot of useful results. The infinitesimals were mysterious, but very convenient: for instance, the chain rule is written

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx},$$

which was thought of as cancelling of fractions of infinitesimals!