This quiz has 5 questions of equal value.

1. Find the antiderivative $F(x)$ of $f(x) = 6x^2 + 4x + 1$ which satisfies $F(1) = 10$.

   Solution: an antiderivative $F(x)$ of $f(x)$ has the general form

   $$F(x) = \int 6x^2 + 4x + 1dx = 2x^3 + 2x^2 + x + C,$$

   where C is some constant. To determine the constant C, we substitute the initial conditions:

   $$F(1) = 2 + 2 + 1 + C = 5 + C = 10,$$

   since $F(1) = 10$. Thus $C = 5$ and

   $$F(x) = 2x^3 + 2x^2 + x + 5.$$

2. Evaluate

   $$\int_1^4 5(x-1)^2 dx.$$

   Solution: the easiest way to do this is by substitution. Let $u = x - 1$, then $du = dx$. When $x = 1$, $u = 0$; when $x = 4$, $u = 3$. Thus

   $$\int_1^4 5(x-1)^2 dx = 5\int_0^3 u^2 d = 5\left.\frac{u^3}{3}\right|_0^3 = 5 \cdot 9 = 45.$$

3. Find the indefinite integral:

   $$\int \frac{x}{\sqrt{x-1}} dx$$

   Solution: another integral by substitution. Let $u = x - 1$, then $du = dx$, and

   $$\begin{aligned}\int \frac{x}{\sqrt{x-1}} dx &= \int \frac{u+1}{\sqrt{u}} du \\ &= \int \frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}} du \\ &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C.\end{aligned}$$

4. Using the left Riemann sum and 3 equal intervals, approximate

   $$\int_1^7 f(x) dx,$$

   where $f(x)$ is a function whose values are given in the table below:

x	0	1	2	3	4	5	6	7	8
$f(x)$	-3	0	1	4	5	12	16	20	100

Solution: by definition, we are computing a sum over the intervals $[1,3]$,$[3,5]$,$[5,7]$, evaluating the function at the left endpoints of ther interval. So, the approximation is

$$\sum_{k=0}^{2} f(1+2k)\cdot 2 = (f(1)+f(3)+f(5))\cdot 2 = (0+4+12)\cdot 2 = 32.$$

5. Compute the following integral:

$$\int_0^{\sqrt{\pi}} 2x\sin\left(x^2+\frac{\pi}{2}\right)dx$$

Another integral by substitution. Letting $u = x^2 + \frac{\pi}{2}$, we have $du = 2xdx$, $u(0) = \frac{\pi}{2}$, $u\left(\sqrt{\pi}\left(= \frac{3\pi}{2}\right.\right.$, so the integral above transforms into

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(u)du = -\cos(u)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0.$$

A note on integrals by substitution

Here is a brief proof of why you can do integrals by substitution the way you are doing them. First, we need to write want we want to prove. Using the standard notation, the substitution rule is written as follows:

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

We use the FTC and the Chain rule to show this. Indeed, let $F(x)$ be the antiderivative of f, so that $F'(x) = f(x)$. Then by the Chain rule,

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x).$$

Therefore, $F(u(x))$ is the antiderivative of the integrand $f(u(x))u'(x)$. Then by the FTC,

$$\begin{aligned}\int_a^b f(u(x))u'(x)dx &= F(u(b)) - F(u(a)) \\ &= \int_{u(a)}^{u(b)} f(u)du,\end{aligned}$$

as wanted. □