This group quiz has 5 questions of equal value.

1. A force of 60 Newtons (about a dozen pounds of force) stretches the rubber band of a slingshot to $x = 0.05$m (about 2 inches) from its resting position. You load a 0.01kg ball into the slingshot and stretch the band to $x = 0.15$m from its resting position. Assuming all energy spent stretching is converted to kinetic energy ($mv^2/2$), what is the muzzle velocity of the ball?

   Solution: we treat the rubber band as a spring. The force required to stretch a spring x units from its resting position is $F(x) = kx$, where k is a constant (called the spring constant). In our problem, we have $0.05k = 60N$, so $k = 1200\frac{N}{m}$.

   The work requider to stretch a spring from $x_0 = 0$ to $x_1 = 0.15m$ is then

$$\int_{x_0}^{x_1} F(x)\ dx = \int_{x_0}^{x_1} kx\ dx = \frac{1}{2}kx^2\Big|_{x_0}^{x_1} = \frac{1}{2}kx_1^2.$$

   This work is the potential energy of the rubber band before it is released. Assuming all this energy is converted to the kinetic energy of the ball of mass $m = 0.01kg$ gives us the equation

$$\frac{1}{2}mv^2 = \frac{1}{2}kx_1^2,$$

   which simplifies to

$$v = x_1\sqrt{\frac{k}{m}}.$$

   Note that from this equation, it follows that **velocity of the ball is directly proportional to how much you stretch the string**. Plugging in numbers (and noting that $1N = 1kg \cdot m/s^2$), we get

$$v = 0.15m\sqrt{\frac{1200N/m}{0.01kg}} = 0.15\sqrt{12 \cdot 100 \cdot 100\frac{kg \cdot m}{s^2 \cdot kg \cdot m}} = 15m\sqrt{12\frac{1}{s^2}} = 30\sqrt{3}\frac{m}{s} \approx 52\frac{m}{s}.$$

   In such problems, units are useful to check your work.

2. Sometimes a glass is rather half-empty than half-full.

   The shape of the inside of an 8-inch tall beer glass in Armadillo Wings can be approximated by a solid of revolution obtained by rotating the curve $x = (y/8)^2$ around the line $x = -1$. How much beer is left afer drinking half the glass by height (that is, up to 4 in mark) as a percentage of the full glass?

   Solution: we compute the volume by slicing with planes perpendicular to the y axis (see Figure 1). The slices are disks of radius $R(y) = x(y) + 1 = (y/8)^2 + 1$. The volume

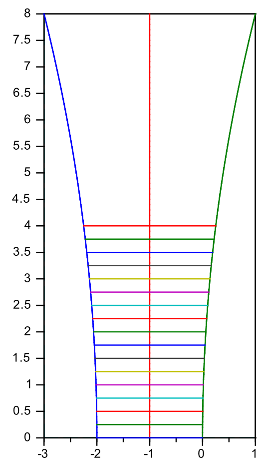


Figure 1: The glass is about 1/3-full.

of the slice at y is $\pi R(y)^2$. The volume V_L of the liquid - which fills the glass from $y = 0$ to $y = 4$ is then

$$\begin{aligned} V_L = \int_0^4 \pi R(y)^2 dy &= \int_0^4 \pi \left(\left(\frac{y}{8}\right)^2 + 1 \right)^2 dy \\ &= \int_0^4 \pi \left(\left(\frac{y}{8}\right)^4 + 2\left(\frac{y}{8}\right)^2 + 1 \right) dy \\ &= \pi \left(\frac{8}{5}\left(\frac{y}{8}\right)^5 + \frac{16}{3}\left(\frac{y}{8}\right)^3 + y \right) \Big|_0^4 \\ &= \pi \left(\frac{8}{5}\left(\frac{4}{8}\right)^5 + \frac{16}{3}\left(\frac{4}{8}\right)^3 + 4 \right). \end{aligned}$$

and the volume V_G of the glass is the same integral taken from 0 to 8. The ratio of the two is

$$\frac{\frac{8}{5}\left(\frac{4}{8}\right)^5 + \frac{16}{3}\left(\frac{4}{8}\right)^3 + 4}{\frac{8}{5} + \frac{16}{3} + 8} = \frac{1}{32} \cdot \frac{24 + 80 \cdot 4 + 4 \cdot 15 \cdot 32}{24 + 80 + 8 \cdot 15} = \frac{3 + 40 + 60 \cdot 4}{224 \cdot 4} = \frac{283}{896} \approx 31.58\%.$$

Note that the arithmetic above is not that hard; however, if with a calculator all we need to do is evaluate

$$\frac{\int_0^4 \pi \left(\left(\frac{y}{8}\right)^2 + 1 \right)^2 dy}{\int_0^8 \pi \left(\left(\frac{y}{8}\right)^2 + 1 \right)^2 dy} \approx 0.3158.$$

For the following problems, **do not evaluate the integral**.

3. Zombie apocalypse strikes, and your best friend Zed turns into a zombie. You find him drinking gasoline from your motorcycle gas tank through a pipe, which sticks 0.4m above the top of the tank.

   Given that the tank is a sphere of diameter of 0.6m and the fuel fills it to the 0.45m mark, write the integral to compute how much energy you will spend pumping it into your tank. Write ρ for the density of the fuel.

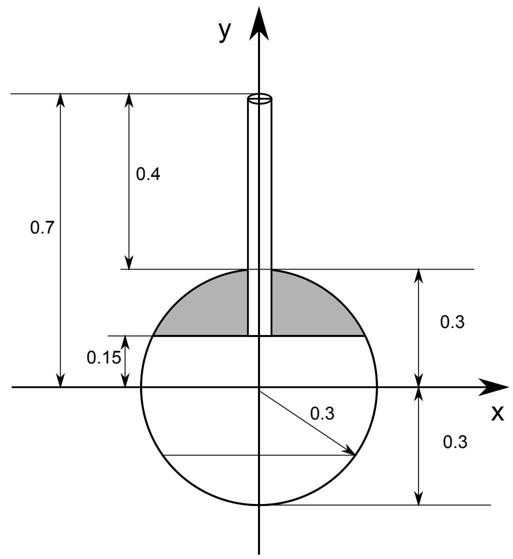


Figure 2: The slices are perpendicular to the y axis.

Solution: we put the origin of the coordinate system at the center of the sphere, as in Figure 2, with y axis pointing upwards. The slices are, again, disks. The points on the sphere in the xy-plane satisfy the equation $x^2 + y^2 = 0.3^2$, so the radius of the disk at depth y is $\sqrt{0.3^2 - y^2}$, and its area is

$$A(y) = \pi\left(\sqrt{0.3^2 - y^2}\right)^2 = \pi\left(0.3^2 - y^2\right).$$

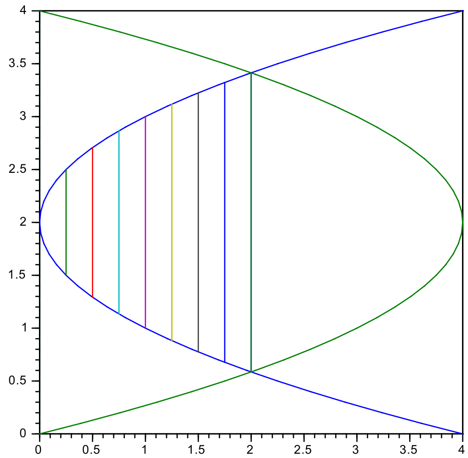
The disk is at depth y, but the distance from $y = 0$ to the end of the pipe is 0.7, and is decreasing as y increases. So the work required to lift a disk at depth y is

$$W(y) = \rho g A(y)(0.7 - y) = \pi\rho g(0.3^2 - y^2)(0.7 - y).$$

Finally, the integral that computes the work is

$$\int_{-0.3}^{0.15} W(y)\, dy = \int_{-0.3}^{0.15} \pi\rho g(0.3^2 - y^2)(0.7 - y)\, dy.$$

4. The area bounded by curves $x = (y-2)^2$ and $x = -(y-2)^2 + 4$ is rotated around $y = -1$. Write the integral which computes the volume of the obtained solid using the washers method.

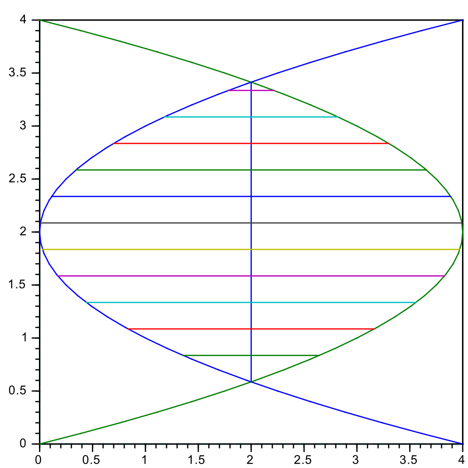


Solution: the slices are washers perpendicular to the x axis. The body is symmetric across the plane perpendicular to the x axis at $x = 2$, so we take the integral from 0 to 2 and then double to get the answer.

To get the radius of the disk, write y as a function of x. For x between 0 and 2, the area is bounded by $y = 2 \pm \sqrt{x}$. The volume is thus

$$2 \int_0^2 \pi \left((2 + \sqrt{x})^2 - (2 - \sqrt{x})^2 \right) \, dx = \frac{64\sqrt{2}}{3}\pi.$$

5. Write the integral to compute the volume of the same solid using the shells method.



Solution: the slices are parallel to the x axis, and turn into cylinders when revolved around it.

Solve the equation $(y-2)^2 = -(y-2)^2 + 4$ to get $y = 2 \pm \sqrt{2}$. Thus our solid lies between $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$, which are the bounds of integration.

The cylindrical shells are obtained by revolving the segment at height y going from $x = (y-2)^2$ to $x = 4 - (y-2)^2$ around the x axis. This segment has length

$$h(y) = \Big(4 - (y-2)^2\Big) - \Big((y-2)^2\Big) = 4 - 2(y-2)^2,$$

which is the height of the cylindrical shell. The radius of the shell is $r(y) = y$, and its surface area is

$$A(y) = 2\pi r(y)h(y) = 2\pi y(4 - 2(y-2)^2).$$

The integral that gives the volume is therefore

$$\int_{2-\sqrt{2}}^{2+\sqrt{2}} 2\pi y(4 - 2(y-2)^2)\; dy = \frac{64\sqrt{2}}{3}\pi.$$