Compute the following integrals:

1. We make the substitution $x = \sin(\theta)$. Then $dx = \cos(\theta)\ d\theta$, and

$$\begin{aligned}
\int \sqrt{1-x^2}\ dx &= \int \sqrt{1-\sin^2(\theta)}\cos(\theta)\ d\theta \\
&= \int \cos^2(\theta)\ d\theta \\
&= \int \frac{1}{2}\left(\cos(2\theta)+1\right)\ d\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\sin(2\theta)+\theta\right)+C \\
&= \frac{1}{2}\left(\frac{1}{2}(2\sin(\theta)\cos(\theta))+\theta\right)+C \\
&= \frac{1}{2}\left(x\sqrt{1-x^2}+\arcsin(\theta)\right)+C.
\end{aligned}$$

2. Here are two solutions to this problem. One way is by noticing that $d(x^3+x) = (3x^2+1)\ dx$, which motivates to break up the fraction as follows:

$$\begin{aligned}
\int \frac{3x^2+x+1}{x^3+x}\ dx &= \int \frac{3x^2+1}{x^3+x} + \frac{x}{x^3+x}\ dx \\
&= \int \frac{3x^2+1}{x^3+x} + \frac{1}{x^2+1}\ dx \\
&= \ln(x^3+x) + \arctan(x) + C.
\end{aligned}$$

Another would be to explicitly solve for a partial fraction decomposition. Write $x^3+x = x(x^2+1)$ as a product of irreducibles; then

$$\begin{aligned}
\frac{3x^2+x+1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\
\frac{3x^2+x+1}{x^3+x} &= \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)} \\
\frac{3x^2+x+1}{x^3+x} &= \frac{(A+B)x^2+Cx+A}{x(x^2+1)} \\
\Rightarrow 3x^2+x+1 &= (A+B)x^2+Cx+A,
\end{aligned}$$

so $A = C = 1$, and $A + B = 3 \Rightarrow B = 2$. We now compute the integral:

$$\begin{aligned}\int \frac{3x^2 + x + 1}{x^3 + x}\,dx &= \int \frac{1}{x} + \frac{2x+1}{x^2+1}\,dx \\ &= \int \frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= \ln(x) + \ln(x^2+1) + \arctan(x) + C.\end{aligned}$$

Of course, $\ln(x) + \ln(x^2+1) = \ln(x^3 + x)$. Note that we don't gain much using partial fractions in this case.

3. Consider the following integral:

$$\int_0^1 \frac{1}{x^a}\,dx$$

For the following values of a, determine whether the integral is convergent or not. If convergent, compute the integral.

$$a = 3, 2, 1, 0, \frac{1}{2}.$$

Solution: for $a = 1$, the integral diverges:

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx \\ &= \lim_{t\to 0^+} \ln(x)\Big|_t^1 \\ &= \lim_{t\to 0^+} (\ln(1) - \ln(t)) \\ &= \infty.\end{aligned}$$

For $a \neq 1$, we can compute the integral using the power rule, then substitute:

$$\begin{aligned}\int_0^1 \frac{1}{x^a}\,dx &= \lim_{t\to 0^+} \int_t^1 x^{-a}\,dx \\ &= \lim_{t\to 0^+} \frac{1}{1-a} x^{1-a}\Big|_t^1\,dx \\ &= \lim_{t\to 0^+} \frac{1}{1-a} - \frac{1}{(1-a)} t^{1-a}\,dx \\ &= \frac{1}{1-a} - \frac{1}{1-a} \lim_{t\to 0^+} t^{1-a}\,dx.\end{aligned}$$

When $a < 1$, the limit on the right-hand side exists and equals 0.

When $a > 1$, the limit on the right-hand side does not exist.

Therefore, the integral diverges for $a > 1$. In particular, it diverges for $a = 2, 3$.

For $a < 1$, the the integral evaluates to $1/1(1-a)$. In particular, for $a = 1/2$, the value is 2, and for $a = 0$, the value is 1.

Note: in case you were absolutely stuck, one thing to notice was that $1/x^0 = 1/1 = 1$, and $\int_0^1 1\ dx = 1$.