Series: for the following series, determine (with justification) whether the given series converges or diverges

- $\sum_{k=7}^{\infty}(-1)^k \frac{k}{\log k}$ **diverges** by the divergence test: $k > \log k > 1$ if $k > 7$, so

$$\lim_{k\to\infty}(-1)^k \frac{k}{\log k} \neq 0.$$

- $\sum_{k=7}^{\infty}(-1)^k \frac{\log k}{k}$ **converges** by the alternating series test:

$$\lim_{k\to\infty}\frac{\log k}{k} = \lim_{k\to\infty}\frac{1/k}{1} = 0.$$

- $\sum_{k=7}^{\infty}\frac{k!}{k^k}$ **converges**: several tests may apply; here's how to do it by the ratio test:

$$\lim_{k\to\infty}\frac{(k+1)!}{(k+1)^{k+1}}\bigg/\frac{k!}{k^k} = \lim_{k\to\infty}\frac{(k+1)k^k}{(k+1)^{k+1}} = \lim_{k\to\infty}\frac{k^k}{(k+1)^k} = \lim_{k\to\infty}\frac{1}{(1+1/k)^k} = 1/e < 1.$$

- $\sum_{k=7}^{\infty}(-1)^k \frac{k!}{k^k}$ **converges** because it converges absolutely (by the previous part). The alternating series test also applies, as

$$0 < \frac{k!}{k^k} = \frac{k}{k}\cdot\frac{k-1}{k}\cdot\frac{k-2}{k}\cdot\ldots\cdot\frac{1}{k} < \frac{1}{k},$$

and so $\lim_{k\to\infty}\frac{k!}{k^k} = 0$ by the Squeeze Lemma.

- $\sum_{k=7}^{\infty}\frac{\sin(k)}{k^2}$ **converges** since it converges absolutely by the comparison test:

$$0 \leq \left|\frac{\sin(k)}{k^2}\right| = \frac{|\sin(k)|}{k^2} < \frac{1}{k^2},$$

and $\sum_{k=7}^{\infty} 1/k^2$ converges.

- For which values of x does the power series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converge?

Solution: consider the ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\bigg/\frac{x^n}{n}\right| = \lim_{n\to\infty}\left|x\frac{n+1}{n}\right| = |x|.$$

So, by the ratio test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Finally, for $x = 1$ the series diverges by the P-test ($p = 1$), and for $x = -1$ the series is alternating and converges by the alternating series test: $\lim_{n\to\infty} \frac{1}{n} = 0$.

So, the series converges for $x \in [-1, 1)$.