1. Determine the center, radius of convergence and the interval of convergence for

$$\sum_{k=1}^{\infty} \frac{2^k(x-1)^k}{7^k k}.$$

Solution: the center is at $x = 1$ (substituting in $x = 1$ makes all terms of this series vanish). To compute the radius of convergence r, use the ratio test to establish that $r = 7/2$:

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k + 1}{a_k} &= \lim_{k\to\infty} \frac{2^{k+1}(x-1)^{k+1}}{7^{k+1}(k+1)} \cdot \frac{7^k k}{2^k(x-1)^k} \\
&= \frac{2}{7}(x-1) \cdot \lim_{k\to\infty} \frac{k+1}{k} \\
&= \frac{2}{7}(x-1).
\end{aligned}$$

By the ratio test, the series diverges when $|2/7(x-1)| > 1$ and converges when $|2/7(x-1)| < 1$, so $|x-1|$ can be at most $7/2$, which is the radius of convergence. Finally, when $x = 7/2$ the series is divergent (it's just the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$, use the integral test), and with $x = -7/2$ the series converges (by alternating series test). Therefore, the interval of convergence is $[1 - 7/2, 1 + 7/2) = [-2.5, 4.5)$.

2. Evaluate the following limit (*Hint: use Taylor series*):

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \underline{\hspace{6cm}}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{x^3} &= \lim_{x\to 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots) - x}{x^3} \\
&= \lim_{x\to 0} -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \ldots \\
&= \lim_{x\to 0} -\frac{1}{3!}.
\end{aligned}$$

3. What does $\sum_{k=0}^{\infty} \frac{\ln(2)^k}{k!}$ converge to?

Solution: note that the above is simply the power series for e^x with $x = \ln(2)$. The answer is, therefore, $e^{\ln(2)} = 2$.

4. Find a power series representation for $\dfrac{x}{(1+x)^2}$.

Solution: one way to solve this is to observe that $\dfrac{-1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1+x}\right)$, and so

$$\begin{aligned}
\frac{x}{(1+x)^2} &= x\frac{d}{dx}\left(-\frac{1}{1-(-x)}\right) \\
&= -x\frac{d}{dx}(1 - x + x^2 - x^3 + \dots) \\
&= -x(-1 + 2x - 3x^2 + \dots) \\
&= (x - 2x^2 + 3x^3 - \dots) \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}kx^k.
\end{aligned}$$

Another way is to simply multiply the terms out:

$$\frac{x}{(1+x)^2} = x \cdot \frac{1}{1+x} \cdot \frac{1}{1+x} = x(1 - x + x^2 - x^3 + \dots)(1 - x + x^2 - x^3 + \dots)$$

(I won't go into details on how to proceed further with this way - figure it out!).

5. Find a 4th degree Taylor polynomial for e^{x^2}. That is, if $e^{x^2} = \sum_{k=0}^{\infty} a_k x^k$ on an interval around 0, write down

$$a_0 + a_1 + a_2x^2 + a_3x^3 + a_4x^4.$$

Solution: $e^u = 1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \dots\dots$. We just plug in $u = x^2$ to obtain

$$\begin{aligned}
e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \dots + \frac{x^{2n}}{n!} + \dots; \\
T_4(x) &= 1 + x^2 + \frac{x^4}{2!}.
\end{aligned}$$

The series we obtained is the Talyor series because when two power series converge to the same function, the series must be equal. Of course, we could calculate the first four derivatives of e^{x^4}, but it would take longer.