1. Find the derivative:

$$\frac{d}{dx}\int_0^x e^{t^2}dt = e^{x^2}.$$

Solution: this holds by the Fundamental theorem of Calculus, which links integration and differentiation.

In plain English, this can be said as follows: **the rate of change of area under the curve is given by its height**.

2. Evaluate:

$$\int_1^4 e^x \cos(e^x)dx = \sin(e^4) - \sin(e)$$

Solution: using u-substitution with $u = e^x$, $du = e^x dx$, obtain

$$\int_1^4 e^x \cos(e^x)dx = \int_e^{e^4} \cos(u)du = \sin(u)|_e^{e^4}.$$

3. Find the indefinite integral:

$$\int \frac{x}{\sqrt{x-1}}dx$$

Solution: another integral by substitution. Let $u = x - 1$, then $du = dx$, and

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}}dx &= \int \frac{u+1}{\sqrt{u}}du \\ &= \int \frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}du \\ &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C.\end{aligned}$$

4. Find the antiderivative $F(x)$ of $f(x) = 6x^2 + 4x + 1$ which satisfies $F(1) = 10$.

Solution: an antiderivative $F(x)$ of $f(x)$ has the general form

$$F(x) = \int 6x^2 + 4x + 1dx = 2x^3 + 2x^2 + x + C,$$

where C is some constant. To determine the constant C, we substitute the initial conditions:

$$F(1) = 2 + 2 + 1 + C = 5 + C = 10,$$

since $F(1) = 10$. Thus $C = 5$ and

$$F(x) = 2x^3 + 2x^2 + x + 5.$$

A note on integrals by substitution

Here is a brief proof of why you can do integrals by substitution the way you are doing them. First, we need to write want we want to prove. Using the standard notation, the substitution rule is written as follows:

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

We use the FTC and the Chain rule to show this. Indeed, let $F(x)$ be the antiderivative of f, so that $F'(x) = f(x)$. Then by the Chain rule,

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x).$$

Therefore, $F(u(x))$ is the antiderivative of the integrand $f(u(x))u'(x)$. Then by the FTC,

$$\begin{aligned}\int_a^b f(u(x))u'(x)dx &= F(u(b)) - F(u(a)) \\ &= \int_{u(a)}^{u(b)} f(u)du,\end{aligned}$$

as wanted. $\square$