The moral of this quiz is that u-substitution can be useful even when you know how to do the integrals without it.
Another theme is using algebra to minimize computational work.

1. Find the area between the curve $y = -(x+4)^2$ and the x-axis for x between -1 and 2.

   Solution: Note that $f(x) = -(x+4)^2$ does not change the sign on the interval $[-1, 2]$ (in fact, it is everywhere negative). Therefore, the following integral computes the area:

$$\int_{-1}^{2} |-(x+4)^2|dx = \int_{-1}^{2} (x+4)^2 dx = \int_{3}^{6} u^2 dx = \frac{1}{3}x^3\Big|_3^6 = 72 - 9 = 63.$$

2. Find the area of the region bounded by the curve $y = (x-3)^2$ and line $y = x - 3$.

   Solution: solving $(x-3)^2 = (x-3)$ gives the solutions $x = 3, 4$. Since $(x-3)^2 < (x-3)$ on $[3, 4]$, the following integral gives the area:

$$\int_{3}^{4} (x-3) - (x-3)^2 dx = \int_{0}^{1} u - u^2 du = \frac{u^2}{2} - \frac{u^3}{3}\Big|_0^1 = \frac{1}{6},$$

   where the integral was computed via the substitution $u = (x-3)$.

   Alternatively, one could translate both graphs by 3 to the left, reducing the problem to $\int_0^1 x^2 - x\ dx$.

3. Find the area bounded by the curve $y = \ln x^2$, the y-axis, and the lines $y = 2$ and $y = 6$.

   Solution: we integrate on the y axis. To do this, we compute x as a function of y:

$$y = \ln x^2 \Rightarrow e^y = x^2 \Rightarrow x = e^{\frac{y}{2}}.$$

   Now $e^{\frac{y}{2}}$ does not change sign for y in $[2, 6]$, so the area is given by

$$\int_{2}^{6} e^{\frac{y}{2}} dy = 2e^{\frac{y}{2}}\Big|_2^6 = 2e^3 - 2e.$$

4. Find the volume of a solid whose base is a circle of radius 3 and the cross-sections perpendicular to the base are squares.

   Solution: a circle of radius 3 can be given by an equation $x^2 + y^2 = 9$. We can pick coordinates so that the x axis runs across the cross-sections (perpendicularly to them), and y axis is parallel to the cross sections.

   The volume of the solid is then given by the integral

$$V = \int_{-3}^{3} A(x)dx,$$

where $A(x)$ gives the cross-section area when the solid is cut by a plane through x perpendicular to the base and parallel to the y axis.

That is, for any x_0, $A(x_0)$ is the area of the square whose side is the intersection of the line $x = x_0$ with the circle $x^2+y^2 = 9$. Solving for y in terms of x_0 gives $y = \pm\sqrt{9 - x_0^2}$, so the side of the square has length $2\sqrt{9 - x_0^2}$, and $A(x_0) = (2\sqrt{9 - x_0^2})^2$. Therefore,

$$A(x) = 4(9 - x^2).$$

Plugging this back into the integral, we obtain

$$V = \int_{-3}^{3} 4(9 - x^2)dx$$

We now use the symmetry to simplify the computation:

$$\int_{-3}^{3} 4(9 - x^2)dx = 2\int_{0}^{3} 4(9 - x^2)dx = 8(9x - \frac{x^3}{3})\Big|_0^3 = 8 \cdot (27 - 9) = 144.$$

Without using the symmetry, you get the same answer, but with a little bit more arithmetic:

$$\int_{-3}^{3} 4(9 - x^2)dx = 4(9x - \frac{x^3}{3})\Big|_{-3}^{3} = 4((27 - 9) - (-27 - (-9))) = 144.$$