1. The region bounded by $y = x^2 + 1$, the x axis, $x = -1$ and $x = 1$, is rotated around the x axis. Find the resulting volume.

   Solution: since the formula for y in terms of x is nice, we want to integrate against x, which means that the slices will run orthogonally to the x axis. When rotated, the slices become disks, i.e. we will use the disks method.

   The radius of the disks will be given by $r(x) = y = x^2 + 1$; the surface area is $A(x) = \pi(r(x))^2$, disk thickness is dx, and the volume of a slice is $dV = A\ dx = \pi(r(x))^2\ dx = \pi(x^2+1)^2\ dx$. The interval over which we integrate is $I = [-1, 1]$, and the total volume is

$$\begin{aligned} V = \int_I dV = \int_{-1}^{1} \pi(x^2+1)^2 dx &= 2\pi \int_0^1 (x^2+1)^2 dx \\ &= 2\pi \int_0^1 x^4 + 2x^2 + 1\ dx \\ &= 2\pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \right) \Big|_0^1 \ = 2\pi \left(\frac{1}{5} + \frac{2}{3} + 1 \right) = \frac{56}{15}\pi. \end{aligned}$$

   (here we used the symmetry to rewrite the integral).

2. The region bounded by $y = x^2 + 1$ and $y = 2$, is rotated around the y axis. Find the resulting volume.

   Solution: since the formular for y in terms of x is nice (it's the same as before), we want to integrate against x again. The region will be sliced orthogonally to the x axis, which means that our slices, when rotated around the y axis, will trace out cylinders. Thus we are using the cylindrical shells method.

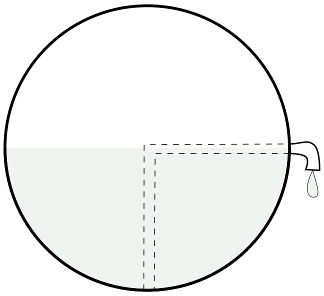
   The height of the cylindrical shell will be given by $h(x) = 2 - y = 2 - (x^2 + 1)$, and the radius by $r(x) = x$. The surface area is $A = 2\pi r(x)h(x) = 2\pi(1 - x^2)x$, shell thickness is dx and shell volume is $dV = A\ dx = 2\pi(x - x^3)\ dx$.

   To find the interval of integration I, we solve for the points of intersection of $y = x^2 + 1$ and $y = 2$. Solving the equation $x^2 + 1 = 2$, we obtain the solutions -1 and 1. Notice that the region is **symmetric** around the y axis, so $I = [0, 1]$, since otherwise we obtain double the volume.

   The volume is given by the integral:

$$\begin{aligned} V = \int_I dV &= \int_0^1 2\pi(x - x^3)\ dx \\ &= 2\pi \int_0^1 (x - x^3)\ dx \\ &= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 \ dx \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}. \end{aligned}$$

3. A spherical tank of radius 1 is half-empty (or half-full, depending on your mood). Calculate the work needed to empty the tank through a spout that runs horizontally to the center of the sphere. Assume a density of ρ and a gravitational constant g.



We slice the body of water so that all water in a slice is at the same gravitaional potential. On the scale of planets, that would mean that our slices would be curved (think why!). On the scale of the problem, luckily, our slices will be flat and parallel to the ground.

We put a coordinate system. Since we are dealing with a sphere, the origin will be in the center to make things easier. Let the x axis be parallel to the ground, and the y axis pointing down.

Since the slices are parallel to the ground, we are integrating against y. The work needed to lift a slice at y is $dW = F \cdot S$, where F is the force of gravity acting on a slice, and S is the distance it needs to go.

We have $S(y) = y$, since we have y pointing down, and $F(y) = mg = \rho g dV$.

The slices are solid disks of thickness dy, whose radius is $r(y) = \sqrt{1 - y^2}$. This is evident from the picture: the points that are as far away as possible from the y axis all lie on the circle of radius 1, which is the largest cross-section of the sphere (with a plane through the origin) and has equation $x^2 + y^2 = 1$.

The area of a slice is $A = \pi r(y)^2 = \pi(1 - y^2)$, the thickness is dy, so the volume is $dV = \pi(1 - y^2)\ dy$. Combining with the above, the work needed to lift it is $dW = \pi \rho g(1 - y^2)y\ dy$.

The interval of integration is $I = [-1, 0]$, since with our coordinate system the bottom of the sphere is at -1 and the middle is at 0.

We integrate to obtain the total work:

$$W = \int_I dW = \int_{-1}^{0} \pi\rho g(1 - y^2)y\ dy = \pi\rho g \left(\frac{y^2}{2} - \frac{y^4}{4} \right) \Big|_{-1}^{0} = \frac{\pi\rho g}{4}.$$