Proving Bennequin's Inequality from Knot Diagrams

Roman Kogan, *SUNY Stony Brook*

Advisor: Olga Plamenevskaya *Department of Mathematics*

Introduction

A contact structure on space is a way of placing planes at each point. It is of interest to mathematicians and physicists, and appears in works on classical mechanics and control theory.

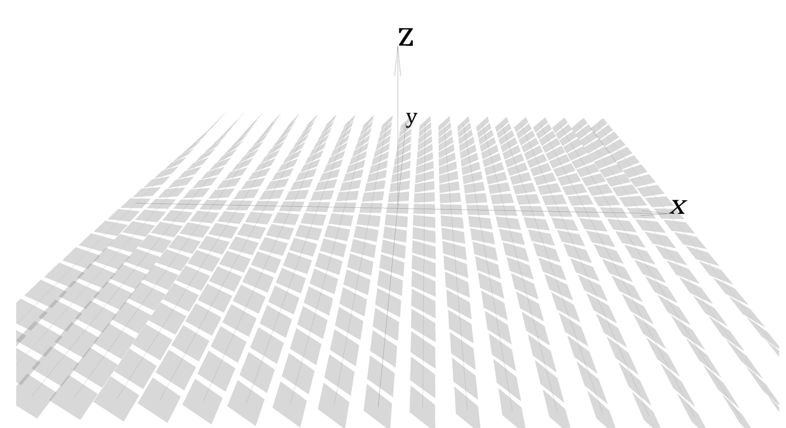
The question of classifying contact structures is closely related to the study of knotted curves in them. D. Bennequin demonstrated the existence of different contact structures on $\mathbf{R}^3$ by demonstrating that for all knots in the standard contact structure a certain inequality holds, and then exhibiting a contact structure with a knot for which the inequality does not hold.

The inequality provides a fundamental relationship between topological (i.e. invariant under non-tearing deformations) and geometric properties of a knot in the standard contact structure. Its proof is quite involved, so the goal of the project was to attempt to utilize planar knot diagrams to come up with a simpler proof.

Contact Structures

ξ_{std}, the *standard contact structure* on the three-dimensional space, is placing a plane spanned by $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z} + y\frac{\partial}{\partial x}\}$ at every point (x, y, z) of $\mathbf{R}^3$.

The slope of the planes only depends on the y coordinate, and this



Such continuously varying placement of planes is called a *plane distribution*, and arises as a kernel of a differential form.

Definition 1 *ξ_{std} is a plane distribution on $\mathbf{R}^3$ given by a kernel of a $1-$ form $\alpha = dz - ydx$ on $\mathbf{R}^3$.*

This is an example of a *totally-non integrable* plane distribution: given any surface, the planes are almost nowhere tangent to it.

For a distribution given by $\ker \alpha$ to have such property, it must satisfy the *Frobenius condition* $\alpha \wedge d\alpha \neq 0$, which is true for $\alpha = dz - ydx$.

A general *contact structure* is any plane distribution that satisfies this condition, and it relates to ξ_{std} as follows:

Theorem 1 *(Darboux): all contact structures are locally diffeomorphic to ξ_{std}.*

That is, all contact structures locally look like the one in the picture: space filled with spiral staircases.

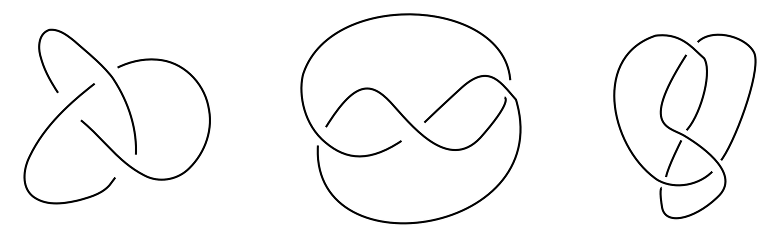
Knots

A mathematical knot knot is a knotted string in space whose ends are welded together. Formally,

Definition 2 *A knot is an embedding of the circle S^1 into $\mathbf{R}^3$.*

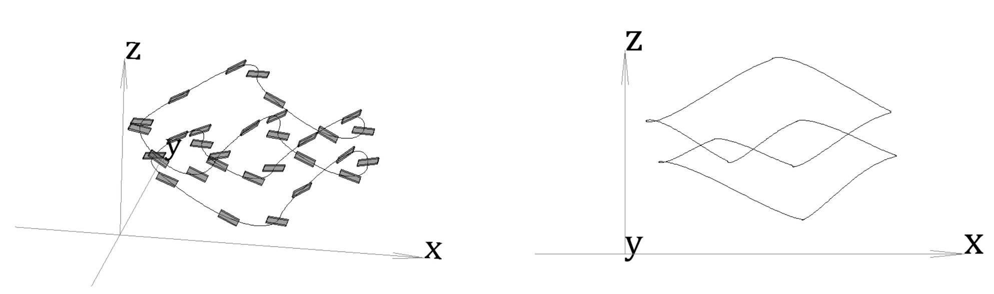
Two knots are not thought to be the same if one can be obtained from another by movement and stretching of the string without tearing or making it cross itself.

Every knot has many *knot diagrams* obtained by projecting the knot on a plane so that the the crossings do not overlap. They look like what you would see if you put the knot on a table. Here are several diagrams of the same trefoil knot:



Definition 3 *A Legendrian Knot is a knot in a contact structure whose tangent vectors lie in corresponding contact planes.*

Here is a Legendrian trefoil and its diagram:



Two Legendrian knots are considered the same if one can be obtained from another by moving the string without tearing (but possibly with stretching) so that it stays tangent to the contact planes at all times.

This is a stricter equivalence condition than the one for general knots, and there exist different Legendrian knots that are the same topological knot. Intuitively, one cannot undo kinks in a Legendrian knot because the space it lives in is kniky.

Legendrian knots have special projections, called the front projection (on the $x - z$ plane, as seen above) and Lagrangian (on the $x - y$ plane). Because tangent vectors satisfy equation $dz - ydx = 0$, we obtain $y = dz/dx$, which forces the front projection to have no vertical tangencies, i.e. in ξ_{std} one has to spiral to go up.

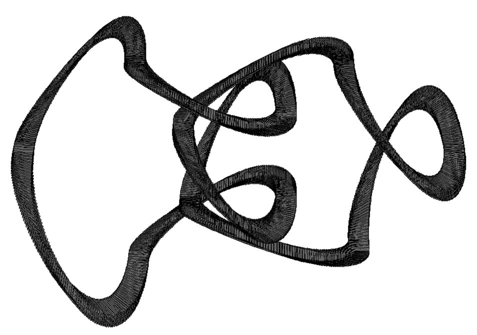
There is no known algorithm that, given two knot diagrams, would tell if they represent the same knot, so the problem of classifying knots is open. That is, mathematicians can not generally tell if two knots are distinct.

However, there are certain numbers, called *knot invariants*, that one can associate with the knot that do not change under the movement of the knot without tearing. That is, they are functions that take knots as input in gives the same value for knots that are equivalent. Knot invariants can be generally computed from the diagram, and help tell knots apart: knots with different values of the invariant must be different.

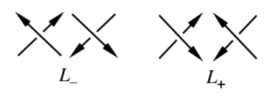
Invariants

The two classical invariants of a Legendrian knot L are the Thurston-Bennequin number, $tb(L)$ and rotation number, $r(L)$.

$tb(L)$ counts how many times ξ_{std} twists around L. If you make a strip that goes along L and is tangent to the contact planes along L, then $tb(L)$ counts the number of twists in the strip.

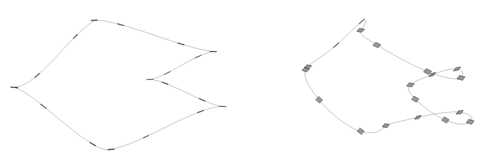


We formally define $tb(L)$ as the number of times L' intersects the Seifert surface of L (see below), where L' is a push-off of L in the z direction. This number measures how many crossings one has to undo to unlink L from L'. It can be computed from oriented knot front diagram: $tb(L) = n_+ - n_- - \frac{1}{2}$no. of cusps. In the equation, n_+ and n_- denote the number of positive and negative crossings, resp., as below:



One can always decrease $tb(L)$ by adding more cusps and preseving the topological knot type, but increasing is not always possible.

If you see the space as the spiral staircase, $r(L)$ measures how many flights L is tall. One can measure it from the $x - y$ projection (it is the rotation number of the curve), or from the front projection (orient the knot, obtain difference between the number of up- and down-going cusps).



These invariants only make sense for Legendrian knots, and distinguish some different Legendrian knots of the same topological type (e.g. the unknot above is nontrivial).

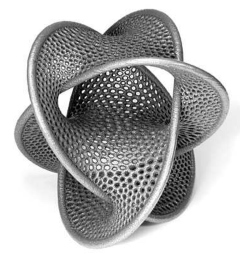
Genus of a knot: there always exist a surface such that the knot forms its boundary. The surface is not unique. One can always get a new one by attaching handles, and thus increasing genus; where genus is a measure of how many "holes" the surface has.

The smallest possible genus for a knot, $g(L)$, is a topological knot invariant (that is very hard to compute).We thus write *Euler characteristic* of the knot L as $\chi(L) = 1 - 2g$.

Seifert's algorithm

Obtains a Seifert surface for a knot or a link

- orient the knot
- resolve the crossing according to the sign given by orientation, obtaining a surface with many boundary components
- join the components with strips twisted to crossing signs



Bennequin's inequality

$$tb(L) + |r(L)| \leq -\chi(L)$$

Given a toplogical knot L, this inequality places an upper bound on $tb(L)$ for all Legendiran knots of the same topological knot type. Since any topological knot can be realized as legendrian, highest value of $tb(L)$ is a topological knot invariant.

Our result

Let Σ be the Seifert surface obtained from a knot diagram $\mathcal{D}$ of L. Then

$$tb(L) + |r(L)| \leq -\chi_{\Sigma}$$

Proof sketch: the Seifet surface obtained with the algorithm is homotopy-equivalent to a graph with V vertices, corresponding to regions, end E edges, corresponding to crossings. Write $E = n_+ + n_-$ as a sum of positive and negative crossings. The inequality reduces to

$$2n_- + min(U, D) \leq -V,$$

where$U$ is number of upward cusps, D - number of downward.

Observe that positive crossings may only occur at a top or bottom of each oriented region. Hence each region must be adjacent to a negative crossing or a cusp (up or down) on its sides (no vertical tangencies). Then $V \leq U + 2n_-$ and $V \leq D + 2n_-$, yielding the result.

Contact information: Roman Kogan, SUNY Stony Brook – Email: *romwell@gmail.com*; Web: *http://www.ic.sunysb.edu/Stu/rkogan*
Acknowledgments: This work was performed during 2009 independent study @SBU The poster was made using the template by Michael Gastpar and Ron Kumon.