Algorithms and Software for Computation in n-dimensional Thompson groups

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Introduction

Thompson groups F, T, and V are interesting objects first studied by Richard Thompson. They are finitely represented infinite groups with some interesting properties. The definition of V naturally generalizes to n dimensions.

Carrying out even basic computations in these groups (finding compositions of maps) by hand is a very tedious process, which necessitated writing a program to do them.

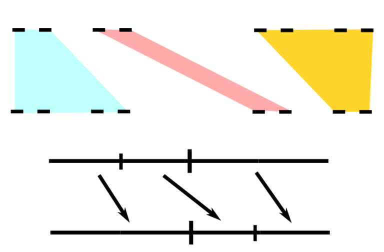
The result is nvTrees - a calculator which can be a very useful tool for anyone who wants to experiment with Thompson groups.

In particular, the software solves the **word problem**.

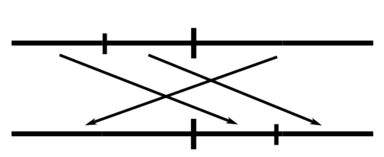
Thompson Groups

Let $\mathbf{C}$ denote the middle-thirds cantor set.

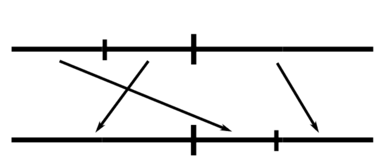
Definition 1 *F is the group of piecewise-linear, order-preserving homeomorphisms of $\mathbf{C}$.*



T is the group of piecewise-linear cyclic order-preserving homeomorphisms of $\mathbf{C}$.



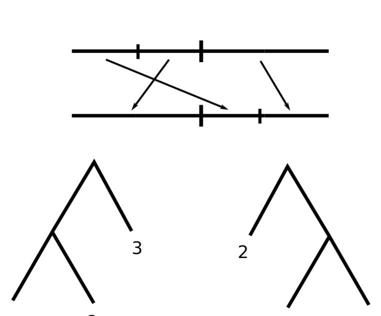
V is the group of piecewise-linear homeomorphisms of $\mathbf{C}$



We extend the definition of V to several dimensions:

Definition 2 *nV is the group of piecewise-linear homeomorphisms of $\mathbf{C}^{\mathbf{n}}$ to itself.*

A useful way to represent maps in V is writing them as labeled tree pairs and a permutation on the leaves:



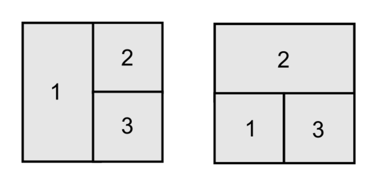
(we don't write the permutation when it is trivial).

Writing elements of nV

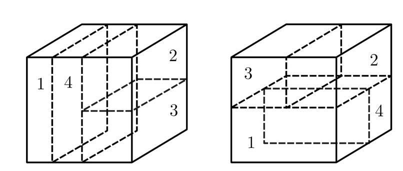
A pattern is a dyadic subdivision of an n-cube.

Maps in nV can be described as labeled pattern pairs.

Example in $2V$:

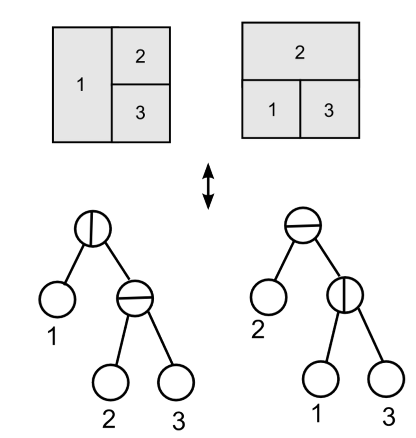


Example in $3V$:



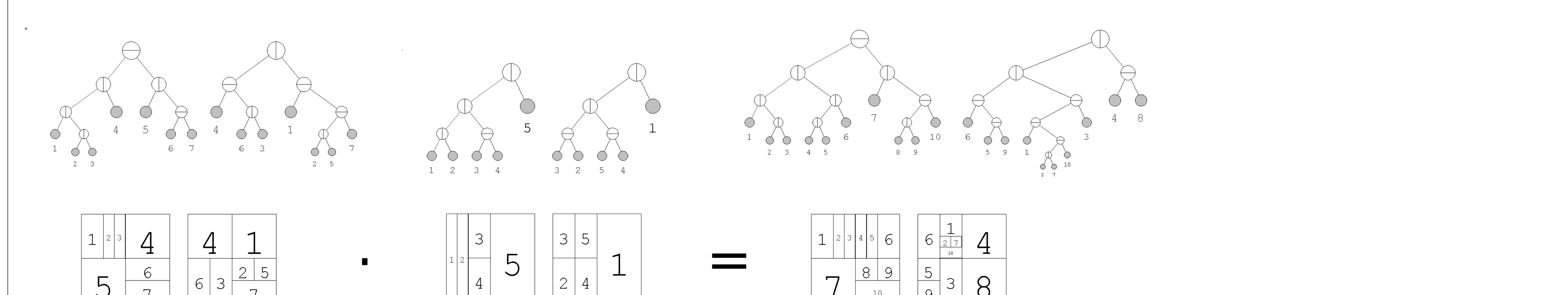
From pattern pairs to labeled trees

Maps in nV can be also written as pairs of labeled trees with permutations:



Color (or label) on the nodes identify the cut axis. To get from trees to patterns, think of the tree as guide on how to cut the square.

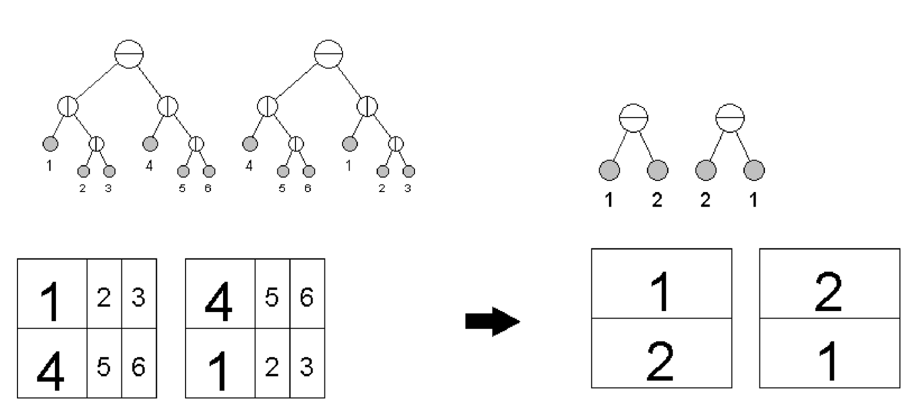
An example of computation with nvTrees



{2110100010200,1201002021000,4 6 3 1 2 5 7 } · {111002000,112002000,3 2 5 4 1 } = {2110100110001021000,1120200220210000200,6 5 9 1 2 7 10 3 4 8 }

Reduced Form

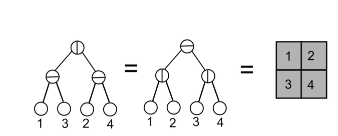
A **reduced form** is a map representation is one where adjacent pairs on the left-hand side are not adjacent on the right. It can be computed in at most quadratic time by a process of removing "exposed carets" (merging the blocks that are moved together).



Word Problem

Reduction on the left solves the **word problem** in at most $O(n^2 \log n)$ time.

For nontrivial elements, this reduced form is not unique due to relations like the one below:



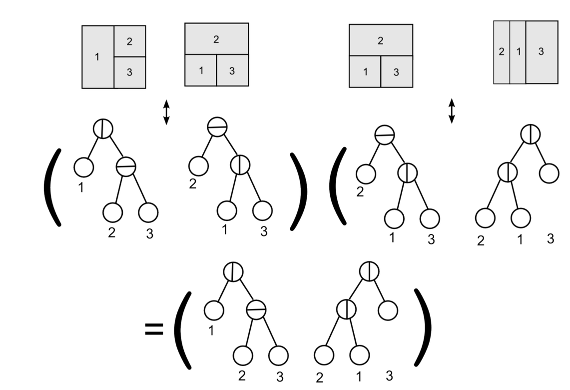
Still, given a preferred order on the labels, one can compute **unique reduced form** in exponential time (by expanding the left tree to a flat tree, and using the relation above to recolor it in a desired way).

Implementation Details

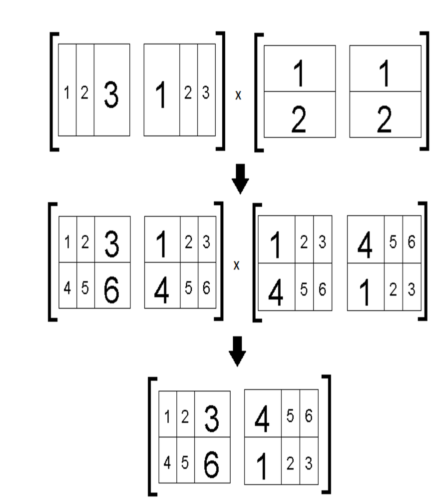
A map is stored using both representations:

- a pair of colored trees and a permutation;
- a pair of patterns, stored as a lists of pattern blocks, and a map on the blocks.

Multiplication of pairs whose inner trees/patterns coincide is just concatenation:



When they don't coincide, the middle patterns are refined to their least common refinement by intersecting blocks with each other (**quadratic** in number of blocks). After this, the maps can be composed in linear time, and the tree representations computed in $O(n^2 \log n)$ (which likely can be improved), and then reduced in linear time.



Theorem 1 *Multiplication of two elements represented by patterns with n blocks may be computed in $O(n^2 \log n)$ time.*

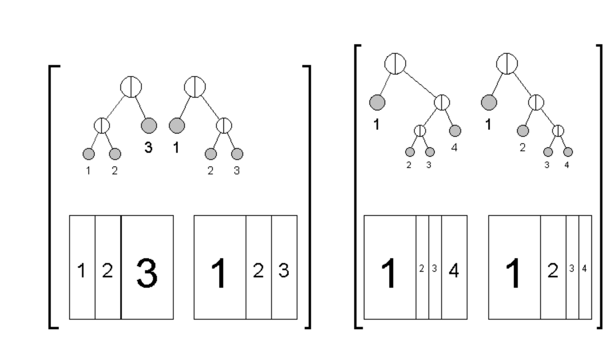
Growth

Since the word problem is solved algorithmically, one can compute the value of the growth function via brute force.

For example, the **growth of Thompson group** F with the standard generators is computed for $n = 1..6$:

n	0	1	2	3	4	5	6
$\gamma(n)$	1	5	17	53	161	475	1381

The generating set used are the following maps and their inverses:



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Parts of software were written during 2008 REU at Cornell University (advisors: Collin Bleak, Francesco Matucci). The poster was made using the template by Michael Gastpar and Ron Kumon.